GEOMETRY OF INTEGRABLE DYNAMICAL SYSTEMS ON 2-DIMENSIONAL SURFACES

NGUYEN TIEN ZUNG AND NGUYEN VAN MINH

ABSTRACT. This paper is devoted to the problem of classification, up to smooth isomorphisms or up to orbital equivalence, of smooth integrable vector fields on 2-dimensional surfaces, under some nondegeneracy conditions. The main continuous invariants involved in this classification are the left equivalence classes of period or monodromy functions, and the cohomology classes of period cocycles, which can be expressed in terms of Puiseux series. We also study the problem of Hamiltonianization of these integrable vector fields by a compatible symplectic or Poisson structure.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper, which is part of our program of systematic study of the geometry and topology of integrable non-Hamiltonian dynamical systems [1, 23, 24, 25, 26, 27], is to describe the local and global invariants and

classification of smooth vector fields on 2-dimensional surfaces, which admit a non-trivial first integral. Such a vector field, together with a first integral, is called an **integrable dynamical system of type (1,1)** (i.e. 1 vector field and 1 function). A special case of systems of type (1,1) is Hamiltonian systems on symplectic surfaces, where the Hamiltonian function itself is a first integral of the Hamiltonian vector field. Invariants of Hamiltonian systems on surfaces have been studied by many people, in particular Fomenko [10] who introduced the notion of "atoms" and "molecules" for semi-local and global topological classification of these systems and systems with $1\frac{1}{2}$ degrees of freedom, and Dufour - Molino -Toulet [7] who gave a symplectic classification in terms of Taylor series of regularized action functions. We will extend the known ideas and results in the Hamiltonian case to the general non-Hamiltonian case.

We will denote an integrable system of type (1,1) by a couple (X, F) or $(X, \mathcal{F})$, where X is a vector field on a 2-dimensional surface Σ such that $X \neq 0$ almost everywhere, F is a first integral of X (i.e. $X(F) = 0$) such that $dF \neq 0$ almost everywhere, and $\mathcal{F}$ is the ring of all first integrals of X. The functional dimension of $\mathcal{F}$ is 1, i.e. $dF \wedge dG = 0$ for any $F, G \in \mathcal{F}$. The main object of our study is X and not F : X is fixed while F can be replaced by any other appropriate first integral. A point $p \in \Sigma$ is called **singular** if it is singular with respect to X, i.e. $X(p) = 0$. The couple $(X, \mathcal{F})$ gives rise to a singular 1-dimensional **associated singular fibration** on the ambient surface Σ: each fiber is a maximal connected subset of Σ on which every first integral $F \in \mathcal{F}$ is constant. Each fiber of this fibration is called a **level set** of $(X, \mathcal{F})$. Notice that the level sets are invariant under the flow of X. A level set $\mathcal{N}$ is called **regular** if there is a first integral $F \in \mathcal{F}$ such that F is regular on $\mathcal{N}$, i.e. $dF \neq 0$ everywhere on $\mathcal{N}$. Remark that a regular level set may contain singular points of X.

We will study integrable systems $(X, \mathcal{F})$ locally, i.e. near a point, semi-locally, i.e. in the neighborhood of a level set, and globally, i.e. on the whole surface. We will describe local and semi-local invariants of these systems, which allow us to classify them up to smooth isomorphisms or smooth orbital equivalence, in the following sense:

Definition 1.1. *Two smooth integrable systems (X_1, F_1) and (X_2, F_2) on two surfaces Σ_1 and Σ_2 respectively are called smoothly* **orbitally equivalent** *if there is a smooth diffeomorphism $\Phi : \Sigma_1 \to \Sigma_2$ such that $\Phi_* X_1 \wedge X_2 = 0$ and the set of singular points of X_2 (where $X_2 = 0$) coincides with the set of singular points of $\Phi_* X_1$. They are called smoothly* **isomorphic** *if $\Phi_* X_1 = X_2$.*

In the above definition, one may replace the smooth (C^∞) category by some other category, e.g. C^1, C^k ($1 \leq k < \infty$), C^ω (analytic). We will work mainly in the smooth and the real analytic categories. In the literature, there are also vector fields which are integrable in a weaker sense: their first

integrals are only smooth outside a small set, see e.g. [11]. Here we require the first integral to be smooth everywhere. Moreover, we will restrict our attention to smooth integrable systems $(X, \mathcal{F})$ which are *weakly nondegenerate* in the sense of Definition 1.2 below. It means that each singular point of $(X, \mathcal{F})$ is either *nondegenerate* (in the sense of [25, 26]), or generic nilpotent (see below). Nondegenerate singular points can be classified into 3 types depending on the eigenvalues of X at them, so in total we allow the following 4 types of singular points in our systems:

(1) Type I. Elliptic: two purely imaginary eigenvalues.
(2) Type II. Hyperbolic with eigenvalue 0: two different real eigenvalues, one of which is 0.
(3) Type III. Hyperbolic without eigenvalue 0: two non-zero real eigenvalues.
(4) Type IV. Generic nilpotent: the linear part of X is nilpotent nonzero, the quadratic part of X is generic and there is a local first integral F with $dF \neq 0$ at the singular point. (See Definition 4.1).

These singularities admit nice local normalizations which linearize the associated singular fibration. For the nondegenerate singularities (Types I, II, III) this fact was shown in [25, 26], and for nilpotent singularities (Type IV) it follows from the definition and Takens-Gong normal form [21, 12]. Moreover, these 4 types of singularities are locally structurally stable, i.e. any C^1-close integrable system will have the same types of singular points.

Definition 1.2. *A smooth integrable system $(X, \mathcal{F})$ on a surface Σ is called* **weakly nondegenerate** *if every of its singular points is of one of the above 4 types I-IV.*
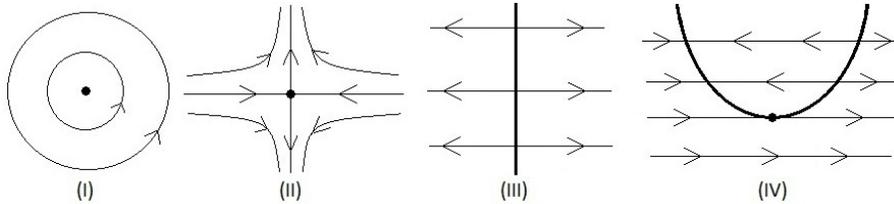
FIGURE 1. Four types of allowed singularities.

We will usually denote by $\mathcal{B}$ the base space of the associated singular fibration of an intgerable system $(X, \mathcal{F})$: each point of $\mathcal{B}$ corresponds to a level set of $(X, \mathcal{F})$. Then we have a natural projection

$$(1.1) \qquad proj : \Sigma \to \mathcal{B}.$$

The topology and the differential structure of $\mathcal{B}$ is induced from Σ via the projection map. The induced differential structure was probably first studied by Reeb and Haefliger [13] for a similar situation. Each first integral $F \in \mathcal{F}$ descends to a smooth function on $\mathcal{B}$. Imitating Dufour - Molino -

Toulet [7], we will view $\mathcal{B}$ as a graph and call it the **Reeb graph** of $(X, \mathcal{F})$: each vertex of the Reeb graph corresponds to a level set which contains at least one singular point of Type I, III or IV. Of course, the Reeb graph is an important orbital invariant of the system, and together with some other discrete invariants it gives an orbital classification of the systems $(X, \mathcal{F})$ similar to a topological clasification obtained by Fomenko and his collaborators (see, e.g., [3, 10, 18]) for integrable Hamiltonian systems with 2 degrees of freedom.

The organization of this paper is as follows: In Section 2 we will describe the local invariants of nondegenerate singular points, using smooth normal forms given by the geometric linearization. We obtain the local classification of these singularities in terms of (left equivalence classes of) the local **period functions** or the **frequency functions**. In Section 3 we give a semi-local classification of nondegenerate level sets in terms of the **monodromy fuctions** or cohomology classes of the **period cocycles**. In Section 4 we study generic nilpotent singularities, where again we obtain a classification in terms of regularized monodromy functions. In Section 5, we put local and semi-local invariants together on the Reeb graph to get a global classification of weakly-nondegenerate integrable systems of type (1,1). Finally, in Section 6, we give necessary and sufficient conditions for an integrable vector field in dimension 2 to be Hamiltonian with respect to some symplectic or Poisson structure.

As will be shown in the paper, our continuous semi-local invariants are similar in nature and constructed in a similar way to the symplectic and flow invariants studied by Dufour - Molino - Toulet [7], Bolsinov [2] and Kruglikov [14, 15] for Hamiltonian or isochore systems on 2-dimensional surfaces, and by Bolsinov, Vũ Ngọc San, and Dullin [2, 4, 9, 22] for higher-dimensional cases. Nevertheless, our invariants are different from and complementary to the invariants in [2, 4, 7, 9, 14, 15, 22], and instead of being expressed in terms of Taylor series as in the cited papers, they are expressed in terms of Puiseux series, due to the non-Hamiltonian nature of the studied systems.

2. Local structure of nondegenerate singularities

2.1. **Elliptic singularities (Type I).** A singular point p (i.e., a point where X vanishes) of an integrable system $(X, \mathcal{F})$ is called a nondegenerate **elliptic** singular point, or also a singular point of **Type I** in this paper, if the eigenvalues of X at p are pure imaginary non-zero (i.e. the linear part of X at p generates a linear circle action on a plane), and there is a first integral $F \in \mathcal{F}$ which is non-flat at p.

Suppose that $(X, \mathcal{F})$ is a smooth integrable system with a singular point p of Type I. According to the local geometric linearization theorem for nondegenerate singularities of integrable systems [25, 26], there is a smooth

coordinate system (x, y) in a neighborhood of p, in which we have the following normal form:

$$(2.1) \qquad X = \frac{2\pi}{f(x^2 + y^2)}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

where f is a smooth function such that $f(0) \neq 0$.

Remark 2.1. Elliptic singular points are also called **centers** in the literature. There is a nice similar normal form result of Maksymenko [16] without the assumption that there is a first integral, but under the assumption that the orbits are closed.

A local coordinate system in which X has normal form like the above form will be called a **canonical coordinate system**. The function f in Formula (2.1) is the local *period function* of the singular point: the orbit through each point (x, y) near p is closed and is of period $f(x^2 + y^2)$. The function $f(x^2 + y^2)$ projects to a smooth function on the local base space $\mathcal{B}_{\mathcal{U}(p)} = proj(\mathcal{U}(p))$, where $\mathcal{U}(p)$ denotes a saturated tubular neighborhood of p and $proj$ is the projection map, which will be denoted by $f_\mathcal{B}$ and also called the local **period function** (on the local base space). In local canonical coordinates, the function $R = x^2 + y^2$ descends to a function on the base space $\mathcal{B}_{\mathcal{U}(p)}$ and becomes a smooth coordinate function there: any other smooth function on $\mathcal{B}_{\mathcal{U}(p)}$ (i.e. whose pull-back by $proj$ is smooth on $\mathcal{U}(p)$) is a smooth function of R. In particular, the period function $f_\mathcal{B}$ is a smooth function on the local base space $\mathcal{B}_{\mathcal{U}(p)}$ and is a function of R: $f_\mathcal{B} = f \circ R$.

It is clear that any singularity of Type I is locally orbitally equivalent to a standard linear center. The period function is a local invariant of the singular point, because any isomorphism will preserve the periods. More precisely, we have:

Proposition 2.2. *Two singularities of Type I of two smooth integrable systems $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ are locally isomorphic if and only if the two corresponding period functions $f_{\mathcal{B}_1}$ and $f_{\mathcal{B}_2}$ on the two associated local base spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are left-equivalent, i.e. there is a local smooth diffeomorphism Φ from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that $f_2 = f_1 \circ \Phi$. In terms of local canonical coordinates, two integrable vector fields*

$$(2.2) \qquad X_1 = \frac{2\pi}{f_1(x_1^2 + y_1^2)}(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1})$$

and

$$(2.3) \qquad X_2 = \frac{2\pi}{f_2(x_2^2 + y_2^2)}(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2})$$

near two singular points of Type I are locally isomorphic if and only if there is a local smooth diffeomorphism ϕ from $(\mathbb{R}_+, 0)$ to itself such that $f_1 = f_2 \circ \phi$.

Proof. It is clear that the left equivalence of the two period functions is a necessary condition for the existence of an isomorphism, because an isomorphism between two vector fields must preserve the periods of periodic orbits.

Conversely, assume that $f_1 = f_2 \circ \phi$ where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a local smooth diffeomorphism. Then it is easy to check that the map

$$(2.4) \qquad (x_1, y_1) \mapsto (x_2, y_2) = \left(\sqrt{\frac{\phi(x_1^2 + y_1^2)}{x_1^2 + y_1^2}} x_1, \sqrt{\frac{\phi(x_1^2 + y_1^2)}{x_1^2 + y_1^2}} y_1 \right)$$

is a smooth map which sends X_1 to X_2. $\qquad\qquad\qquad\qquad\square$

2.2. **Hyperbolic singularities with eigenvalue 0 (Type II).** A singular point $p \in \Sigma$ of a smooth integrable system $(X, \mathcal{F})$ on Σ is called **nondegenerate hyperbolic singular point with eigenvalue 0**, or also a singular point of **Type II** if it satisfies the following conditions:

i) The linear part of X at p has two different real eigenvalues: $\gamma_1 = 0$ and $\gamma_2 \neq 0$.

ii) There is $F \in \mathcal{F}$ which is not flat at p, i.e. its Taylor series is non-trivial.

According to the local geometric linearization theorem [25, 26], for each nondegenerate hyperbolic singular point p with eigenvalue 0, there is a local smooth coordinate system (x, y) in a neighborhood $\mathcal{U}$ of p such that X has the normal form

$$(2.5) \qquad\qquad\qquad X = \gamma(y) x \frac{\partial}{\partial x},$$

where $\gamma(0) \neq 0$, and F is a function of y. In particular, the set of singular points of X near p is a smooth curve $\mathcal{S}$ given by the equation

$$(2.6) \qquad \mathcal{S} = \{ q \in \mathcal{U} \mid X(q) = 0 \} = \{ q \in \mathcal{U} \mid x(q) = 0 \},$$

and for each point $q \in \mathcal{S}$, the eigenvalue of X at q is equal to $\gamma(y(q))$. Since the eigenvalue does not depend on the choice of coordinates, the eigenvalue function on $\mathcal{S}$,

$$(2.7) \qquad\qquad\qquad q \in \mathcal{S} \mapsto f(y(q))$$

is a local invariant of X at p. We can formulate this fact as the following proposition, whose proof is straightforward.

Proposition 2.3. *Two smooth integrable systems $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ are locally smoothly isomorphic near two respective singular points p_1 and p_2 of Type II if and only if there is a local eigenvalue-preserving diffeomorphism from the curve $\mathcal{S}_1$ of singular points of X_1 near p_1 to the curve $\mathcal{S}_2$ of singular points of X_2 near p_2. In terms of local normal forms, $X_1 = \gamma_1(y) x \frac{\partial}{\partial x}$ is locally smoothly isomorphic to $X_2 = \gamma_2(y) x \frac{\partial}{\partial x}$ if and only if γ_1 is left equivalent to γ_2, i.e. there is a local diffeomorphism $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\gamma_1(y) = \gamma_2(\psi(y))$.*

Observe that the function $f(y) = 2\pi\sqrt{-1}/\gamma(y)$, where $\gamma(y)$ is the *eigenvalue function* in the local normal form (2.5) can also be interpreted as a local *period function* as follows: Complexify the system in the coordinate x (so x is now a complex coordinate, and y is still a real coordinate, the manifold has 1 complex dimension plus 1 real dimension). Then the flow

of X on each local complex line $\{y = const\}$ is periodic in imaginary time and is of period equal to $f(y)$. So we will call $f(y)$ the (imaginary) local **period function** of X near a singular point of Type II. Proposition 2.3 can be paraphrased as follows: Type II singular points are classified up to local isomorphisms by the left equivalence class of their imaginary period functions.

2.3. Hyperbolic singularities without eigenvalue 0 (Type III).

To say that p is a singular point of Type III of $(X, \mathcal{F})$ means that X has two non-zero real eigenvalues at p, and there is a first integral which is not flat at p. According to the local geometric linearization theorem [25, 26], X has the following normal form:

$$(2.8) \qquad X = h(x^a y^b)\left(\frac{x}{a}\frac{\partial}{\partial y} - \frac{y}{b}\frac{\partial}{\partial x}\right)$$

where $a, b \in \mathbb{N}$ are coprime, and h is a smooth function such that $h(0) \neq 0$.

In the above canonical coordinates, the function $x^a y^b$ generates the ring local first integrals of X. More precisely, if G is a smooth local first integral, then in each local quadrant $\{\epsilon x \geq 0, \delta y \geq 0\}$, where $\epsilon = \pm$ and $\delta = \pm$, we can write

$$(2.9) \qquad G(x, y) = g^{\epsilon, \delta}(h(x^a y^b))$$

where $g^{+,+}, g^{+,-}, g^{-,+}, g^{-,-}$ are smooth functions which have the same Taylor series at 0. The proof of this fact follows easily from formal computations.

Similarly to the Type II case, X is of (complex) toric degree 1 (see [23] for the notion of toric degree of a vector field at a singular point), and the $\mathbb{T}^1$-action in the complexified space associated to X is generated by the linear vector field

$$(2.10) \qquad Y = 2\pi\sqrt{-1}\left(\frac{x}{a}\frac{\partial}{\partial y} - \frac{y}{b}\frac{\partial}{\partial x}\right)$$

in canonical coordinates. In the analytic case, Y is uniquely determined by X (see [23]), though in the smooth case it is only unique up to a flat term. So any automorphism or isomorphism of X will also be an automorphism or isomorphism of Y up to a flat term. Thus, in order to understand the local automorphisms of X, we need to understand the automorphisms of the linear vector field Y (or $Y/(2\pi\sqrt{-1})$).

The function $h(x^a y^b)$ in the normal form (2.8) is directly related to the local period function of X in the complexified space (at least in the analytic case) : The flow of the vector field $X = h(x^a y^b)\left(\frac{x}{a}\frac{\partial}{\partial y} - \frac{y}{b}\frac{\partial}{\partial x}\right)$ in $\mathbb{C}^2$ in the imaginary time will have closed orbits with periods equal to $\frac{2\pi ab\sqrt{-1}}{h(x^a y^b)}$. We will call $h(x^a y^b)$ the **frequency function** of X near p.

Proposition 2.4. *Let a and b be two coprime natural numbers.*

1) A local real analytic diffeomorphism $(x, y) \mapsto (x_1, y_1)$ preserves the linear vector field $\dfrac{x}{a}\dfrac{\partial}{\partial x} - \dfrac{y}{b}\dfrac{\partial}{\partial y}$ if and only if there are two local analytic

functions ρ_1, ρ_2 of one variable such that $\rho_1(0) \neq 0, \rho_2(0) \neq 0$ and

$$(2.11) \qquad \begin{cases} x_1 = x\rho_1(x^a y^b) \\ y_1 = y\rho_2(x^a y^b) \end{cases}.$$

2) A local smooth diffeomorphism $(x, y) \mapsto (x_1, y_1)$ preserves the linear vector field $\dfrac{x}{a}\dfrac{\partial}{\partial x} - \dfrac{y}{b}\dfrac{\partial}{\partial y}$ if and only if there are four couples of local smooth functions $\rho_1^{\varepsilon,\delta}, \rho_2^{\varepsilon,\delta}$ where $\varepsilon = \pm$ and $\delta = \pm$, which do not vanish at 0, such that the Taylor series of $\rho_1^{\varepsilon,\delta}$ and $\rho_2^{\varepsilon,\delta}$ do not depend on ε and δ, and such that

$$(2.12) \qquad \begin{cases} x_1 = x\rho_1^{\varepsilon,\delta}(x^a y^b) \\ y_1 = y\rho_2^{\varepsilon,\delta}(x^a y^b) \end{cases}$$

if $\varepsilon x \geq 0, \delta y \geq 0$.

Proof. 1) Let $(x, y) \mapsto (x_1, y_1)$ be a local real analytic diffeomorphism which preserves the linear vector field $X^{(1)} = \frac{x}{a}\frac{\partial}{\partial x} - \frac{y}{b}\frac{\partial}{\partial y}$. Since $\{x = 0\}$ (resp. $\{x = 0\}$) is the stable (resp. unstable) manifold of $X^{(1)}$, we must have $x_1 = 0$ on $\{x = 0\}$ and $y_1 = 0$ on $\{y = 0\}$. In other words, we can write $x_1 = x.\theta_1(x, y), y_1 = y.\theta_2(x, y)$ where θ_1, θ_2 are two analytic functions. Note that the time-t flow of the linear vector field $X^{(1)}$ multiplies x by $e^{t/a}$, and also multiplies x_1 by $e^{t/a}$ because the map $(x, y) \mapsto (x_1, y_1)$ preserves $X^{(1)}$. Therefore the quotient $\frac{x_1}{x} = \theta_1(x, y)$ is invariant by the flow of $X^{(1)}$. In other words, $\theta_1(x, y)$ is a first integral of $X^{(1)}$. Any local analytic first integral $\theta_1(x, y) = \rho_1(x^a y^b)$. Similarly, we have $\theta_2(x, y) = \rho_2(x^a y^b)$.

Conversely, it is easy to see that the map $(x, y) \mapsto (x\rho_1(x^a y^b), y\rho_2(x^a y^b))$ preserves $X^{(1)} = \frac{x}{a}\frac{\partial}{\partial x} - \frac{y}{b}\frac{\partial}{\partial y}$.

2) The proof in the smooth case is similar, except that we must write $\theta_1(x, y) = \rho_1^{\varepsilon,\delta}$ dependent on the quadrant in $\mathbb{R}^2$, where $\rho_1^{+,+}, \rho_1^{+,-}, \rho_1^{-,+}$ and $\rho_1^{-,-}$ are four functions which have the same Taylor series at 0. (Actually, the number of functions can be reduced from 4 to 2, because, for example, if a is odd then $\rho_1^{+,\delta}$ and $\rho_1^{-,\delta}$ can be chosen to be the same function). $\qquad\square$

Proposition 2.5. *Let a and b be two coprime natural numbers.*

1) Two local real analytic vector fields $X_1 = h_1(x_1^a y_1^b)\left(\frac{x_1}{a}\frac{\partial}{\partial x_1} - \frac{y_1}{b}\frac{\partial}{\partial y_1}\right)$ and $X_h = h_2(x_2^a y_2^b)\left(\frac{x_2}{a}\frac{\partial}{\partial x_2} - \frac{y_2}{b}\frac{\partial}{\partial y_2}\right)$ are locally analytically isomorphic if and only if h_1 and h_2 are left equivalent, i.e. we can write $h_1 = h_2 \circ \psi$ where $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is analytic with $\psi'(0) \neq 0$.

2) In the smooth case, when h_1 and h_2 are smooth functions, then X_1 and X_2 are locally smoothly isomorphic if and only if h_1 and h_2 are formally left-equivalent, i.e.

$$(2.13) \qquad Taylor(h_1) = Taylor(h_2) \circ \psi$$

where ψ is a formal series with $\psi(0) = 0, \psi'(0) \neq 0$, and $Taylor(h)$ means the Taylor series of h.

Proof. 1) The toric degree of X_1 (in the sense of [23]) is 1, and the corresponding $\mathbb{T}^1$-action in the complexified space which preserves X is generated by $\frac{x_1}{a}\frac{\partial}{\partial x_1} - \frac{y_1}{b}\frac{\partial}{\partial y_1}$. Thus if there is a local analytic diffeomorphism Φ such that $\Phi_* X_1 = X_2$, then we also have $\Phi_*\left(\frac{x_1}{a}\frac{\partial}{\partial x_1} - \frac{y_1}{b}\frac{\partial}{\partial y_1}\right) = \left(\frac{x_2}{a}\frac{\partial}{\partial x_2} - \frac{y_2}{b}\frac{\partial}{\partial y_2}\right)$. According to Proposition 2.4, after the diffeomorphism Φ we can write

$$(2.14) \qquad \begin{cases} x_2 = x_1\rho(x^a y^b) \\ y_2 = y_1\theta(x^a y^b) \end{cases}.$$

In particular, $x_2^a y_2^b = (x_1^a y_1^b).\rho(x_1^a y_1^b).\theta(x_1^a y_1^b) = \psi(x_1^a y_1^b)$ where $\psi(z) = z\rho(z)\theta(z)$. We also have $\Phi_* h_1 = h_2$, therefore $h_2 = h_1 \circ \psi$.

Conversely, assume that $h_2 = h_1 \circ \psi$. Then it is easy to check that the map $(x_2, y_2) = \Phi(x_1, y_1)$ given by the formula

$$(2.15) \qquad \begin{cases} x_2 = x_1 \\ y_2 = y_1\sqrt[b]{\dfrac{\psi(x_1^a y_1^b)}{x_1^a y_1^b}} \end{cases}$$

(this map is well-defined because $\frac{\psi(z)}{z} \neq 0$ when $z = 0$) sends X_1 to X_2.

2) The "only part" in the smooth case is absolutely similar to the analytic case. The "if" part follows from Sternberg-Chen's theorem [5, 20] which says that if two smooth hyperbolic vector fields are formally isomorphic then they are locally smoothly isomorphic. $\qquad\square$

3. Semi-local structure of nondegenerate singularities

3.1. **Level sets with singular points of Type II.** Denote by $\mathcal{N}$ a level set of a weakly nondegenerate smooth integrable system $(X, \mathcal{F})$ on a compact surface Σ, which contains a singular point of Type II, but does not contain singular points of the other Type I, III, IV. As will be shown below, $\mathcal{N}$ is a smooth circle. A tubular neighborhood of $\mathcal{N}$ will be either orientable (a cylinder) or non-orientable (a Mobius band). The case when it is a Mobius band will be called the **twisted case**, and the case when it is a cylinder will be called the **non-twisted case**. Notice that if Σ is orientable then we only have the non-twisted case.

Proposition 3.1. *With the above notations and assumptions, we have:*
i) $\mathcal{N}$ *is a smooth circle.*
ii) $\mathcal{N}$ *contains an even number* $2m > 0$ *of singular points of Type II.*
iii) In the non-twisted case there is a tubular neighborhood $\mathcal{U}(\mathcal{N})$ *of* $\mathcal{N}$ *such that the projection map*

$$(3.1) \qquad\qquad proj : \mathcal{U}(\mathcal{N}) \to \mathcal{B}$$

from $\mathcal{U}(\mathcal{N})$ *to the base space* $\mathcal{B}$ *of the associated singular fibration is a smooth trivial circle fibration over its image*

$$(3.2) \qquad\qquad \mathcal{B}_{\mathcal{U}(\mathcal{N})} := proj(\mathcal{U}(\mathcal{N})),$$

and the set of singular points $\mathcal{S} = \{q \in \mathcal{U}(\mathcal{N}) \mid X(q) = 0\}$ *is a disjoint union*

(3.3)
$$\mathcal{S} = \bigsqcup_{i=1}^{2m} \mathcal{S}_i$$

of $2m$ *smooth sections of the circle fibration* $\mathcal{U}(\mathcal{N}) \to \mathcal{B}_{\mathcal{U}(\mathcal{N})}$, *and every point of* $\mathcal{S}$ *is singular of Type II.*

iv) In the twisted case, there is still a tubular neighborhood $\mathcal{U}(\mathcal{N})$ *of* $\mathcal{N}$ *which is saturated and smoothly foliated by the level sets of* $(X, \mathcal{F})$, *and* $\mathcal{N}$ *is the only exceptional leaf of this foliation in* $\mathcal{U}(\mathcal{N})$ *(the one with non-trivial holonomy), the set of singular points* $\mathcal{S} = \{q \in \mathcal{U}(\mathcal{N}) \mid X(q) = 0\}$ *is still a disjoint union* $\mathcal{S} = \bigsqcup_{i=1}^{2m} \mathcal{S}_i$ *of an even number of smooth curves.*

v) In the non-twisted case, $\mathcal{N}$ *is a regular level set of the associated fibration, i.e. there is* $F \in \mathcal{F}$ *which is regular at* $\mathcal{N}$. *In the twisted case,* $\mathcal{N}$ *is a singular level set of Morse-Bott type, i.e. there is* $F \in \mathcal{F}$ *such that* F *is of the type* $F = z^2$ *in the neighborhood of every point of* $\mathcal{N}$.

Proof. i) Denote by $\mathcal{K}$ smallest closed invariant set of the vector field X which contains p and satisfies the following additional property: If $q \in \mathcal{K}$ and $\mathcal{O}$ is an orbit of X such that its closure contains q, then $\mathcal{O} \subset \mathcal{K}$. (Such a set $\mathcal{K}$ exists because it is the intersection of all closed invariant sets which satisfy these properties). It follows immediately from the definition of $\mathcal{K}$ that $\mathcal{K}$ is connected. By continuity, any first integral $F \in \mathcal{F}$ is constant on $\mathcal{K}$, so we have $\mathcal{K} \subset \mathcal{N}$.

We will show that in fact $\mathcal{K}$ is a smooth circle and $\mathcal{N} = \mathcal{K}$.

Since $\mathcal{K} \subset \mathcal{N}$, all singular points of X in $\mathcal{K}$ are of type II by our assumptions. Remark that, if q is a singular point of type II, then there are exactly two regular orbits of X which contain q in their closure, and moreover their union forms together with q a curve which is smooth at q: in local normal form $X = f(y)x\frac{\partial}{\partial x}$ they are given by $\{x < 0, y = 0\}$ and $\{x > 0, y = 0\}$. Remark also that the number of singular points in $\mathcal{K}$ is finite, otherwise there would exist an accumulation point $q_\infty \in \mathcal{K}, q_\infty = \lim_{n\to\infty} q_n, q_n \neq q_\infty, q_n \in \mathcal{K}$ singular of Type II, and then any smooth first integral will be constant on $\{q_n, n \in \mathbb{N}\} \subset \mathcal{K}$, which implies that it is flat at q_∞, which is a contradiction to our definition of type II singular points.

Starting from the point p, take the two regular orbits whose closures contain p, then take the singular points in the closure of these orbits, then take the regular orbits whose closures contain these new singular points, and so on. By definition, all these orbits and singular points belong to $\mathcal{K}$. The process must stop after a finite number of steps, because $\mathcal{K}$ contains a finite number of singular points. This process gives us a smooth curve, so $\mathcal{K}$ is closed and smooth, i.e. it is a circle.

By our definition of Type II singularities, there is a smooth first integral F, which is constant on $\mathcal{K}$ of course, and which is not flat at p. It follows

easily that any point outside of $\mathcal{K}$ can be separated from $\mathcal{K}$ by a first integral. In other words, $\mathcal{K}$ is a level set, i.e. $\mathcal{K} = \mathcal{N}$, and therefore $\mathcal{N}$ is a smooth circle.

ii) This statement is a particular simple case of our study of nondegenerate $\mathbb{R}^n$-actions on n-manifolds in [27] (see Subsection 3.2 of [27]).
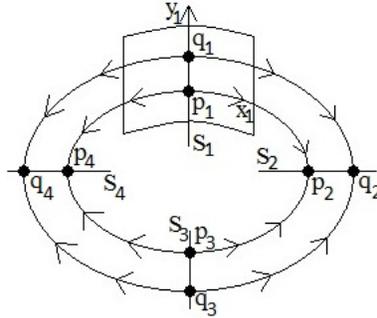

FIGURE 2. Trivial holonomy.

iii) Consider the non-twisted case. Denote by $p_1 = p, p_2, \ldots, p_{2m}$ the singular points of Type II of X on $\mathcal{K}$ in a cyclic order. Near each p_i fix a canonical coordinate system (x_i, y_i) in a neighborhood $\mathcal{U}_i$ of p_i in which $X = f_i(y_i)x_i\frac{\partial}{\partial x_i}$, and such that these coordinate systems give the same orientation on $\mathcal{K}$ and in a neighborhood of $\mathcal{K}$. Denote by $\mathcal{S}_i = \mathcal{U}_i \cap \{x_i = 0\}$ the curve of singular points of Type II passing through p_i.

Take a point $q_1 \in \mathcal{S}_1, q_1 \neq p_1$, q_1 close enough to p_1. Then the set $\{y_1 = y_1(q_1), x_1 > 0\}$ is a local regular orbit which tends to q_1 in one direction. By continuity, this orbit remains close to $\mathcal{K}$ and enters a small neighborhood $\mathcal{U}_2$ of p_2 in the other direction, and so it tends to some point $q_2 \in \mathcal{S}_2$ close to p_2. Similarly, there is a regular orbit of X which tends to q_2 in one direction and tends to a point $q_3 \in \mathcal{S}_3$ in the other direction, and so on. Finally, there is a regular orbit which tends to $q_{2m} \in S_{2m}$ in one direction, and tends to a point $q_1' \in \mathcal{S}_1$ in the other direction. So we get a smooth map from $\mathcal{S}_1$ to itself defined by $q_1 \mapsto q_1'$, which may be called the **holonomy** of X along $\mathcal{K}$. Remark that, in the non-twisted case, $y_i(q_i)$ have the same sign for all i, and in particular $y_1(q_1)$ and $y_1(q_1')$ have the same sign. Due to the existence of a smooth global first integral which is non-flat for every singular point, it is easy to see that the holonomy is actually trivial, i.e. $q_1 = q_1'$. Indeed, if $y_1(q_1) > y_1(q_1') > 0$ for example, then we can repeat this process (iterate the holonomy map) to get a sequence of points $q_1, q_1', q_1'', q_1''', \ldots, q_1^{(n)}, \ldots$, which must tend to some point $q_1^\infty \in \mathcal{S}_1$, and any global smooth integral will be flat at q_1^∞, which contradicts our assumptions about the singular points of $(X, \mathcal{F})$. (In this proposition, we really need the existence of global non-flat first integrals, and not just local non-flat first integrals, otherwise there will be counter-examples). Since the holonomy is

trivial, we have a smooth foliation of a neighborhood of $\mathcal{N}$ into circles with trivial holonomy. The rest of the proof is straightforward.

iv) The proof of Assertion iv) is similar to the proof of Assertion iii).

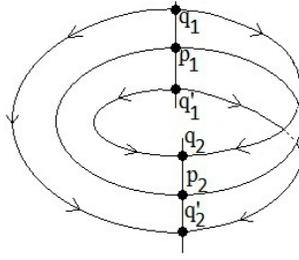v) The proof is straightforward. □

FIGURE 3. Mobius band in the twisted case.

Observe that, on each curve $\mathcal{S}_i$ of singular points in a tubular neighborhood $\mathcal{U}(\mathcal{N})$ of $\mathcal{N}$, we have an eigenvalue function, whose value at each point $q \in \mathcal{S}_i$ is the non-zero eigenvalue of X at q. In the non-twisted case, these functions may be viewed as functions on the local base space $\mathcal{B}_{\mathcal{U}(\mathcal{N})} = proj(\mathcal{U}(\mathcal{N}))$ via the projection map, which we will denote by $E_1, \ldots, E_{2m}$:

For $\xi \in \mathcal{B}_{\mathcal{U}(\mathcal{N})}, E_i(\xi)$ is the eigenvalue of X at $proj^{-1}(\xi) \cap \mathcal{S}_i$.

Of course, these eigenvalue functions on $\mathcal{B}_{\mathcal{U}(\mathcal{N})}$ are invariants of X in a neighborhood $\mathcal{U}(\mathcal{N})$ of $\mathcal{N}$. In the twisted case, we still have $2m$ eigenvalue functions, but they don't descend to functions on $\mathcal{B}_{\mathcal{U}(\mathcal{N})}$ in general, only to a branched 2-covering of $\mathcal{B}_{\mathcal{U}(\mathcal{N})}$.

Besides the eigenvalue functions, which are invariant of local character of the vector field X, there is another invariant of X in $\mathcal{U}(\mathcal{N})$, of a more global character, which was discovered in [27] and called the **monodromy**. The monodromy is defined as follows:

Let M be an closed invariant curve (level set) in $\mathcal{U}(\mathcal{N})$, and denote by $q_1, \ldots, q_{2n}$ the singular points of Type II on M in a cyclic order. (if $\mathcal{U}(\mathcal{N})$ is twisted and $M \neq \mathcal{N}$ then $n = 2m$, otherwise $n = m$). As was observed in [27], for each q_i there is an involution (reflection) which preserves X and exchanges the regular orbit on the left of q_i with the orbit on the right of q_i. Take a regular point z_0 which lies between q_{2n} and q_1, and put $z_1 = \sigma_1(z_0), z_2 = \sigma_2(z_1), \ldots, z_{2m} = \sigma_{2m}(z_{2m-1})$. Then z_{2m} and z_0 lies on the same orbit of X, and there is a unique number $\mu = \mu(M)$ such that

$$(3.4) \qquad\qquad \varphi_X^\mu(z_0) = z_{2m}.$$

It was observed in [27] that μ depends only on X, M and the choice of the orientation on M: when the orientation is inversed the μ changes to $-\mu$.

Since μ can be defined for each level set M in $\mathcal{U}(\mathcal{N})$, we get a map from $\mathcal{B}_{\mathcal{U}(\mathcal{N})}$ to $\mathbb{R}$, which associates to each element $\xi \in \mathcal{B}_{\mathcal{U}(\mathcal{N})}$ the monodromy of X on $proj^{-1}(\xi)$ (with respect to a given choice of orientation).

Proposition 3.2. *The topological type of $\mathcal{U}(\mathcal{N})$ (twisted or non-twisted), the number of singular points of Type II on $\mathcal{N}$, the eigenvalue functions, and the monodromy function form together a complete set of invariants of a weakly nondegenerate smooth integrable system $(X, \mathcal{F})$ in the neighborhood of a compact level set $\mathcal{N}$ which contains only singular points of Type II.*
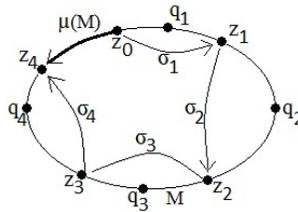

FIGURE 4. Monodromy

It means that $(X_1, \mathcal{F}_1)$ in $\mathcal{U}(\mathcal{N}_1)$ is semi-locally isomorphic to $(X_2, \mathcal{F}_2)$ in $\mathcal{U}(\mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are level sets of Type II, if and only if there is a semi-local fibration-preserving diffeomorphism Φ from $\mathcal{U}(\mathcal{N}_1)$ to $\mathcal{U}(\mathcal{N}_2)$ which is bijection between the sets of Type II singular points, and which also preserve the monodromy function and the eigenvalue functions.

Proof. The proof in the non-twisted case is just a parametrized version of a result of [27] (see Theorem 3.7 of [27]). The twisted case can be reduced to the non-twisted case by taking a double covering. $\square$

Remark 3.3. Proposition 3.2 remains true when $\mathcal{N}$ does not contain any singular point of X at all, and in that case the monodromy function is nothing but the period function, i.e. the time it takes for the flow to go full circle on closed orbits.

3.2. Level sets with singular points of Type III.

Definition 3.4. *We will say that a level set $\mathcal{N}$ of a smooth integrable system $(X, \mathcal{F})$ is a **singular level set of Type III** if $\mathcal{N}$ contains at least one singular point of Type III, and any singular point of X in $\mathcal{N}$ is of Type II or Type III.*

Proposition 3.5. *Let $\mathcal{N}$ be a singular level set of Type III of integrable system $(X, \mathcal{F})$ on a compact surface Σ. Then we have:*
i) Topologically, $\mathcal{N}$ is a connected finite graph, whose edges are regular orbits of X, and whose vertices are singular points of Type II or Type III: Each singular point of Type II is a vertex of valency 2, and each singular point of Type III is a vertex of valency 4. Moreover, $\mathcal{N}$ is a finite union

$\bigcup_{k=1}^{s} \mathcal{S}_k$ of s *smooth circles* $\mathcal{S}_k$ $(s \geq 1)$ *with transversal (self-)intersections at singular points of Type III.*

ii) There exists a smooth first integral $F \in \mathcal{F}$ *such that* $F = 0$ *on* $\mathcal{N}$ *and the multiplicity of* F *at each* $\mathcal{S}_k$ *is a natural number* m_k, *such that the greatest common divisor of* $(m_1, \ldots, m_s)$ *is 1 or 2, and such that any other first integral* $G \in \mathcal{F}$ *can be written as*

$$(3.5) \qquad\qquad G = g(F) + G_{flat},$$

where g *is a smooth function and* G_{flat} *is flat at* $\mathcal{N}$. *If* $gcd(m_1, \ldots, m_s) = 2$, *then* F *can be chosen to be a non-negative function.*

Proof. i) The proof of Assertion i) is similar to the proof of Assertion i) of Proposition 3.1.

ii) Similarly to the proof of Assertion iii) of Proposition 3.1, take a point q on a local curve L which intersects $\mathcal{N}$ transversally at a regular point q_0, and let it go by the flow of X and jump over the points of Type II whenever the flow tends to such a point. Then we will get a path which moves along some components of $\mathcal{N}$ and then returns to a point q' on L after going around. In the non-twisted case, when $\mathcal{U}(\mathcal{N})$ is orientable, then q and q' lie on the same side with respect to q_0 on L. In the twisted case, the first return point q' may lie on the opposite side of q with respect to q_0 on L. If q' lies on the opposite side of q, then we continue the path until we return to L again, this time at a point $q"$ which lies on the same side as q. Using arguments similar to the ones in the proof of Proposition 3.1, one can show that this return map must in fact be the identity map, i.e. we have either $q = q'$ or $q = q"$.

It follows from the above arguments that $\mathcal{U}(\mathcal{N}) \setminus \mathcal{N}$ is foliated by smooth invariant circles, and these circles are also regular level sets of $(X, \mathcal{F})$.

Take an arbitrary smooth first integral $G \in \mathcal{F}$ which is non-flat at $\mathcal{N}$. Then $G(\mathcal{S}_k) = 0$ for each circle $\mathcal{S}_k$ in $\mathcal{N}$, and moreover the multiplicity (i.e. order of vanishing) of G on $\mathcal{S}_k$ is a finite number $M_k < \infty$.

If $p_i \in \mathcal{S}_j \cap \mathcal{S}_k$ with a local canonical coordinate system (x_{ij}, x_{ik}) such that

$$(3.6) \qquad\qquad X = f_i(x_{ij}^{\alpha_{ij}} x_{ik}^{\alpha_{ik}}) \left(\frac{x_{ij}}{\alpha_{ij}} \partial x_{ij} - \frac{x_{ik}}{\alpha_{ik}} \partial x_{ik} \right),$$

and $\alpha_{ij}, \alpha_{ik} \in \mathbb{N}, gcd(\alpha_{ij}, \alpha_{ik}) = 1$, then the smoothness of G at p_i implies that

$$(3.7) \qquad\qquad \frac{M_k}{\alpha_{ik}} = \frac{M_j}{\alpha_{ij}} \in \mathbb{N}.$$

Notice that, if $\frac{M_k}{\alpha_{ik}} = \frac{M_j}{\alpha_{ij}}$ then it automatically means that this fraction is a natural number, because α_{ij} and α_{ik} are coprime. Put

$$(3.8) \qquad\qquad m_k = \frac{M_k}{D},$$

where $D = gcd(M_1, \ldots, M_s)$ is the greatest common divisor of $M_1, \ldots, M_s$. Then we still have $\frac{m_k}{\alpha_{ik}} = \frac{m_j}{\alpha_{ij}} \in \mathbb{N}$ for every singular point p_i of Type II ($p_i \in \mathcal{S}_j \cap \mathcal{S}_k$). It is easy to see that the function

$$(3.9) \qquad \sqrt[D]{G^2}$$

is a well defined smooth first integral whose order of vanishing on $\mathcal{S}_k$ is $2m_k$ for all $k = 1, \ldots, s$. Either this function, or its square root $\sqrt[D]{G}$ if a smooth single-valued function $\sqrt[D]{G}$ can be defined in a neighborhood of N, will be the required first integral. $\qquad \square$

Remark 3.6. In the above proposition, even in the non-twisted (i.e. when $\mathcal{U}(N)$ is orientable) we may have $gcd(m_1, \ldots, m_s) = 2$, while even in the twisted case we have have $gcd(m_1, \ldots, m_s) = 1$, as the examples pictured in Figure 5 show.
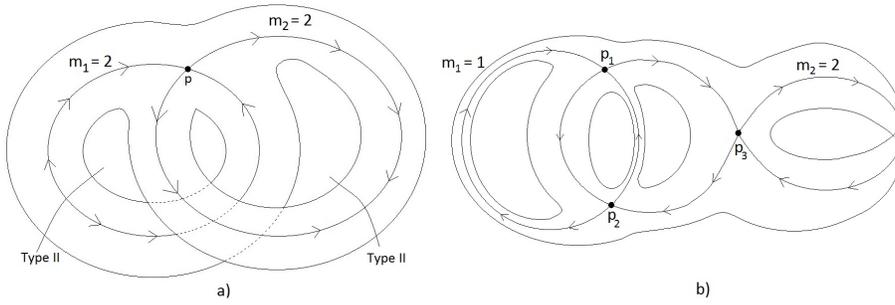

FIGURE 5. a) Non-twisted example with $gcd(m_1, \ldots, m_s) = 2$; b) Twisted example with $gcd(m_1, \ldots, m_s) = 1$

The semi-local first integral function F given by the above proposition can be viewed as a function on the local base space $\mathcal{B}_{\mathcal{U}(N)}$, and will be called a local **coordinate function** on $\mathcal{B}_{\mathcal{U}(N)}$: if G is any other smooth function on $\mathcal{B}_{\mathcal{U}(N)}$ then there exists a family of smooth real functions g_i of one variable, one for each local edge of $\mathcal{B}_{\mathcal{U}(N)}$ (when $\mathcal{B}_{\mathcal{U}(N)}$ is a local graph with one vertex and more than one edges), such that $G = g_i(F)$ on each edge, and all the functions g_i have the same Taylor expansion (i.e. the diffrence of any two of them is a flat function).

Proposition 3.7. *Two smooth integrable systems $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ near two respective singular level sets of Type III N_1 and N_2 are semi-locally orbitally equivalent if and only if there is a homeomorphism from a neighborhood of N_1 to a neighborhood of N_2, which sends N_1 to N_2, singular points of Type II on N_1 to singular points of Type II on N_2, and preserves the ratio $-a : b$ of the two eigenvalues of each singular point of Type III.*

Proof. It follows easily from Proposition 3.5. $\qquad \square$

3.3. **The period cocycle.** In order to classify singular level sets of Type
III semi-locally, we need an additional invariant, which is the cohomology
class of the period cocycle defined below. The construction is similar to
the one introduced in [7] for obtaining symplectic invariants of Hamiltonian
systems, though our situation is more complicated.

Let us fix a semilocal first integral F in $\mathcal{U}(N)$ with lowest multiplicity,
as given by Assertion ii) of Proposition 3.5. According to Proposition 2.5,
for each singular point $p_i \in N$ of Type III, we can choose a local canonical
coordinate system (x_i, y_i) in which the vector field X has the form

$$(3.10) \qquad X = h_i(x_i^{a_i} y_i^{b_i})\left(\frac{x_i}{a_i}\frac{\partial}{\partial y_i} - \frac{y_i}{b_i}\frac{\partial}{\partial x_i}\right)$$

Take an edge E in N. For simplicity, let us assume for the moment that
E does not contain singular points of Type II. To fix the notations, assume
that p_i (resp. p_j) is the limit of the points of E by the flow of the vector
field X in the negative (resp. positive) time direction, and that near p_i we
have $E \supset \{y_i = 0, x_i > 0\}$ and near p_j we have $E \supset \{y_j = 0, x_j < 0\}$.
Denote by $A_E = \{x_i = 1\}$ and $B_E = \{x_j = -1\}$ the two local curves (in two
local coordinate systems) given by these equations. Then the flow of X will
take each point of A_E to a point of B_E after some time. So we get a time
function for going from a point of A_E to B_E by X, which may be viewed as
a function on A_E (this function may admit any value, positive or negative).
Since the multiplicity of F at E is equal to $m_E = m_{i-} = m_{j-}$, time function
we can view this function as a function of F, which is a-priori not regular
in F but regular in F^{1/m_E}. (In the analytic case, it would be a Puiseux
series in F; in the smooth case we can still talk about its Puiseux series).
Let us denote this local function of one variable by P_E, i.e. the value of the
function at each point $q \in A_E$ is $P_E(F(q))$. If E contains points of Type
II, then P_E can be defined in the same way, by jumping over the points of
Type II, like we did in [27] and in the previous subsection for the definition
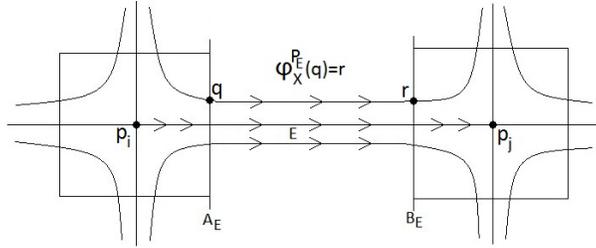of monodromy.

FIGURE 6. Component P_E of the period cocycle

Thus, for each E we get a function P_E. The family $(P_E, \ E$ is an edge of $N)$
will be called the **period cocycle**. It is easy to see that this period cocy-
cle is arbitrary, i.e. any family of local functions $(P_E, \ E$ is an edge of $N)$,

where each $P_E(z)$ is a smooth function in z^{1/m_E}, can be realized by a smooth integrable system, by the gluing method. However, this cocycle is not an invariant of the system, because it depends on the choice of F and of local canonical coordinates. In order to get an invariant, we have to take its equivalence class with respect to a natural equivalence relation generated by 2 kinds of operations: changing a canonical coordinate system by another canonical coordinate system, and changing the first integral F by another first integral. Changing F by another first integral (which has the same multiplicity at the components of N as F) simply leads to the left equivalence (in the sense of left equivalence of maps), while changing local coordinates leads to a cohomology class. So our invariant is the left equivalence class of a cohomology class.

Assume, for example, that the coordinate system (x_1, y_1) in a neighborhood of p_1 is replaced by another canonical coordinate system (x'_1, y'_1) in the same neighborhood. According to Proposition 2.5, we have

$$(3.11) \qquad x'_1 = x_1 \rho^{\pm}(x_1^{a_1} y_1^{b_1}), y'_1 = y_1 \theta^{\pm}(x_1^{a_1} y_1^{b_1})$$

for some *multi-branched* smooth functions $\rho^{\pm}, \theta^{\pm}$ such that $\rho(0) \neq 0, \theta(0) \neq 0$. Here, a multi-branched smooth function is a finite family of functions (for example ρ^+ and ρ^-) which coincide up to a flat term at the point in question, i.e. all the branches have the same Taylor series (so that they can be glued together to become a smooth function on a non-separated manifold or a Reeb graph). Denote by $E_1 = \{y = 0, x > 0\}, E_2 = \{y = 0, x < 0\}, E_3 = \{y > 0, x = 0\}, E_4 = \{y < 0, x = 0\}$ the four local edges of N having p_1 as a vertex. Then the functions $P_{E_1}, \ldots, P_{E_4}$ will be changed by the following rule under the above change of coordinates:

$$(3.12) \qquad \begin{cases} P'_{E_1}(F) &=& P_{E_1}(F) + \xi^+(F) \\ P'_{E_2}(F) &=& P_{E_2}(F) + \xi^-(F) \\ P'_{E_3}(F) &=& P_{E_3}(F) + \zeta^+(F) \\ P'_{E_4}(F) &=& P_{E_4}(F) + \zeta^-(F) \end{cases}$$

where $\xi^+(F)$ and $\xi^-(F)$ are multi-branched functions which are smooth in F^{1/k_1} and have the same formal expansion, where k_1 is the natural number such that F has the same multiplicity at the local edges near p_1 as the function $x_1^{k_1 a_1} y_1^{k_1 b_1}$ (i.e. the multiplicity m_{E_1} of F at E_1 is equal to $k_1 b_1$ and so on), and the same holds true for $\zeta^+(F)$ and $\zeta^-(F)$. Besides the fact that $\xi^{\pm}(F)$ and $\zeta^{\pm}(F)$ must be regular in F^{1/k_1}, they can be chosen arbitrarily (i.e. we can choose the corresponding multi-branched functions $\rho^{\pm}$ and $\theta^{\pm}$ in order to get the desired functions ξ and ζ). In other words, we have a **coboundary** of the type

$$(3.13) \qquad \begin{cases} P_{E_1} &=& \xi^+(F) &=& \hat{\xi}^+(F^{1/k_1}) \\ P_{E_2} &=& \xi^-(F) &=& \hat{\xi}^-(F^{1/k_1}) \\ P_{E_3} &=& \zeta^+(F) &=& \hat{\zeta}^+(F^{1/k_1}) \\ P_{E_4} &=& \zeta^-(F) &=& \hat{\zeta}^-(F^{1/k_1}) \end{cases}$$

($\hat{\xi}^{\pm}$ and $\hat{\zeta}^{\pm}$ are smooth functions; the other components are zero), and these are the generators of the space of coboundaries which can be obtained by changes of coordinates.

Remark 3.8. A-priori, each component P_E of the cocycle is only regular in F^{1/m_E}, why the coboundaries have a higher level of regularity (each "elementary" coboundary is regular in some F^{1/k_i}). Due to this fact, if $P_E \neq 0$ for some edge E we cannot in general substract from our cocycle a coboundary so that P_E becomes 0.

Definition 3.9. *The class of the period cocycle (P_E, E is an edge of N) in the quotient space of the linear space of all period cocycles by the linear space of all coboundaries (generated by the coboundaries given by Formula (3.13)) is called its* **cohomology class**.

Similarly to [7, 22], since the coboundaries can be multi-branched, we can use them to kill all the flat terms in the cocycles, i.e. any two cocycles which are the same up to a flat term are cohomologic. Thus the cohomological class of the cocycle (P_E, E is an edge of N) depends only on its asymptotic expansion. This asymptotic expansion is a (multi-dimensional) **Puiseux series** in F (where F is viewed as a local coordinate function on the Reeb graph $\mathcal{B}$). The cohomology class itself can be expressed in terms of a Puiseux series with a family of coefficients equal to 0 (those which can be eliminated by a normalizing coboundary).

The frequency functions $h_i(x_i^{a_i} y_i^{b_i})$ in Formula (3.10) near p_i will also be viewed as functions of F. Similarly to the peiod cocycle, these frequency functions are not regular in F, but regular in a fractional power of F, and they are determined by the system only up to a flat term, so we will also retain only the Puiseux series of these functions in F.

Remark 3.10. In complex analysis there is a problem of multi-valuedness of Puiseux series, but we don't have this problem here with our systems on real manifolds, because by $F^{1/k}$ of course we mean the unique real k-th root of F if k is odd, and if k is even (in which case F must also be positive) we usually mean the unique positive real root.

Theorem 3.11. *Let $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ be two smooth integrable systems with two respective nondegenerate singular level sets N_1 and N_2 of type III. Then these two systems are semi-locally isomorphic, i.e. there is a smooth diffeomorphism from a neighborhood of N_1 to a neighborhood of N_2 which sends X_1 to X_2 if and only if they are semi-locally orbitally equivalent and satisfy the following additional condition: there is a local coordinate function F_1 (resp. F_2) on the local base space $\mathcal{B}_{\mathcal{U}(N_1)}$ (resp. $\mathcal{B}_{\mathcal{U}(N_2)}$) such that the Puiseux series in F_1 of the frequency functions and the cohomology class of the period cocycle of X_1 coincide with the Puiseux series in F_2 of the corresponding frequency functions and the the cohomology class of the period cocycle of X_2.*

Proof. The proof is straightforward, based on the above discussions and the standard gluing and extension methods, like in, e.g., [6, 7, 9, 22]. □

4. Generic nilpotent singularities (Type IV)

4.1. **Local normal form.** Assume that $X(p) = 0$ at some point p, and that the linear part of X at p is non-zero nilpotent, i.e. is has Jordan form $X^{(1)} = y\frac{\partial}{\partial x}$. According to a classical result of Takens [21], there exists a formal coordinate system in which X can be formally written as

$$(4.1) \qquad X = \left(y + x^2 f(x)\right)\frac{\partial}{\partial x} + x^2 g(x)\frac{\partial}{\partial y},$$

where f and g are formal series.

Let us assume, moreover, that $dF(p) \neq 0$ for some $F \in \mathcal{F}$, i.e. F is a regular function at p. Then, according to a theorem of Gong [12], in the above expression of X one can put $g(x) = 0$, i.e. X can be formally written as

$$(4.2) \qquad X = \left(y + x^2 f(x)\right)\frac{\partial}{\partial x},$$

and F is a function of y.

We will say that the nilpotent singularity of $(X, \mathcal{F})$ at p is *generic*, if $f(0) \neq 0$ in the above formal normal form. Another equivalent definition of genericity, without using Takens-Gong normal form, is as follows:

Definition 4.1. *Let p be a nilpotent singular point of an integrable system $(X, \mathcal{F})$. Then we will say that p is a **generic** nilpotent singular point if the following conditions are satisfied:*

i) There exists a local smooth first integral F such that $dF(p) \neq 0$.

ii) There is a local smooth coordinate system (x, y) in which F is a function of y, and

$$(4.3) \qquad \frac{\partial^2}{\partial x^2}X := [\frac{\partial}{\partial x}, [\frac{\partial}{\partial x}, X]] \neq 0.$$

Let p be a generic nilpotent singularity of (X, F). Then there is a coordinate system (x, y) in a neighborhood $\mathcal{U}$ of p such that y is a local first integral of X, and the linear part of X is in Jordan form, i.e.

$$(4.4) \qquad X = \left(y + G(x, y)\right)\frac{\partial}{\partial x},$$

where $G(x, y)$ is a smooth function such that

$$(4.5) \qquad G(0, 0) = \frac{\partial G}{\partial x}(0, 0) = \frac{\partial G}{\partial y}(0, 0) = 0.$$

Moreover, according to our assumptions,

$$(4.6) \qquad \frac{\partial^2 G}{\partial x^2}(0, 0) \neq 0.$$

According to the implicit function theorem, for each x near 0, there is a unique $y = \gamma(x)$ such that

$$(4.7) \qquad G\big(x, \gamma(x)\big) + \gamma(x) = 0.$$

Moreover, the function $x \mapsto \gamma(x)$ is a smooth function such that $\gamma(0) = 0$, $\dfrac{d\gamma}{dx}(0) = 0$ but $\dfrac{d^2\gamma}{dx^2}(0) \neq 0$, i.e. γ has Morse singularity at 0.

The points $(x, y) = \big(x, \gamma(x)\big)$ in the neighborhood $\mathcal{U}$ of p are precisely those points at which X vanishes. By Morse theorem, using a smooth change of coordinates, we may assume that the curve

$$(4.8) \qquad \mathcal{S} = \{X = 0\} \cap \mathcal{U} = \{(x, y) \in \mathcal{U} \mid y = \gamma(x)\}$$

is the standard parabolic curve $\{y = x^2\}$, i.e. $\gamma(x) = x^2$. So we have

$$(4.9) \qquad X = \widetilde{G}(x, y)\frac{\partial}{\partial x},$$

where $\widetilde{G} = y + G(x, y)$ vanishes on the curve $\{y = x^2\}$. It means that $\widetilde{G}$ is divisible by the function $y - x^2$, i.e. we can write

$$(4.10) \qquad X = g(x, y).(y - x^2)\frac{\partial}{\partial x},$$

where $g(x, y)$ is a smooth function such that $g(0, 0) \neq 0$.

Notice that X has eigenvalue equal to

$$(4.11) \qquad\qquad - 2xg(x, x^2)$$

at each point $(x, x^2) \in \mathcal{S}$. The function $x[g(x, x^2) - g(-x, x^2)]$ is an even function in x, and so can be considered a smooth function in x^2 which vanishes at 0. Denote by g_1 a smooth function of one variable such that

$$(4.12) \qquad xg_1(x^2) = [g(x, x^2) - g(-x, x^2)]/2.$$

Similarly, there is a smooth function g_0 such that

$$(4.13) \qquad g_0(x^2) = [g(x, x^2) + g(-x, x^2)]/2.$$

Put

$$(4.14) \qquad X_1 = \big(g_0(y) + xg_1(y)\big)(y - x^2)\frac{\partial}{\partial x}.$$

Then (X_1, y) is also a local smooth integrable system with a generic nilpotent singularity at p, and moreover the set $\{X_1 = 0\}$ locally coincides with the set $\{X = 0\}$, and X_1 has the same eigenvalue as X at every point of this set.

Proposition 4.2. *With the above notations, the vector fields*

$$(4.15) \qquad X = g(x, y).(y - x^2)\frac{\partial}{\partial x}$$

and

$$(4.16) \qquad X_1 = \big(g_0(y) + xg_1(y)\big)(y - x^2)\frac{\partial}{\partial x}$$

are locally smoothly isomorphic. More precisely, there is a smooth local diffeomorphism $\Phi : (\mathcal{U}, 0) \to (\mathcal{U}, 0)$, which preserves the coordinate y and such that $\Phi_* X = X_1$.

Proof. We will use Moser's path method [17]. See Appendix A1 of [8] for an introduction to this method. Take the following path of vector fields:

$$(4.17) \qquad X_t = tX + (1 - t)X_1 = \rho_t(x, y).(y - x^2)\frac{\partial}{\partial x},$$

where $\rho_t = tg + (1 - t)(g_0 + xg_1)$.

The main point is to show the existence of a time-dependent vector field

$$(4.18) \qquad Z_t = \varphi_t(x, y)(y - x^2)\frac{\partial}{\partial x}$$

such that

$$(4.19) \qquad -\mathcal{L}_{Z_t} X_t = [X_t, Z_t] = \frac{d}{dt} X_t = X_1 - X.$$

If Z_t exists, then its time-1 flow will be the required local diffeomorphism which moves X to X_1. Thus we have to solve the equation

$$(4.20) \qquad \left[\rho_t(x, y).(y - x^2)\frac{\partial}{\partial x}, \varphi_t(x, y)(y - x^2)\frac{\partial}{\partial x}\right] = X_1 - X.$$

Notice that $X_1 - X$ vanishes up to the second order on the curve $\mathcal{S} = \{(x, y) \in \mathcal{U} \mid y = x^2\}$, i.e. we can write

$$(4.21) \qquad X_1 - X = h(x, y).(y - x^2)\frac{\partial}{\partial x}.$$

Then Equation (4.20) is equivalent to

$$(4.22) \qquad \rho_t \varphi_t' - \varphi_t \rho_t' = h$$

where the apostrophe means derivation by x. This last equation is equivalent to

$$(4.23) \qquad \left(\frac{\varphi_t}{\rho_t}\right)' = \frac{h}{\rho_t^2},$$

which admits a smooth solution

$$(4.24) \qquad \varphi_t(x, y) = \rho_t(x, y) \int_0^x \frac{h(s, y)}{\rho_t^2(s, y)} ds.$$

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.2 means that any generic nilpotent singularity of a smooth integrable system (X, F) has the following smooth normal form:

$$(4.25) \qquad X = \big(g_0(y) + xg_1(y)\big)(y - x^2)\frac{\partial}{\partial x},$$

where $g_0(0) \neq 0$ (one can fix $g_0(0) = 1$ if one wishes).

Notice that the eigenvalues of the singular point $(x, y) = (x, x^2) = (\pm\sqrt{y}, y)$ of X is

$$(4.26) \qquad \mp 2\sqrt{y}\big(g_0(y) \pm \sqrt{y}g_1(y)\big) = -2x\big(g_0(x^2) + xg_1(x^2)\big).$$

Of course, these eigenvalues depend only on X and on the parametrization of the local singular curve $\mathcal{S} = \{X = 0\}$ by the coordinate x (such that locally $\mathcal{S} = \{(x, x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$).

If $\{(x_1, y_1 = x_1^2) \mid x_1 \in \mathbb{R}\}$ is another parametrization of $\mathcal{S}$ in another smooth coordinate system (x_1, y_1), then there is an odd function ψ such that $\psi(0) = 0, \psi'(0) \neq 0, \psi(-x) = -\psi(x)$ and

$$(4.27) \qquad x_1 = \psi(x), y_1 = \psi(\sqrt{y})^2 = \psi(x)^2 \text{ on } \mathcal{S}.$$

Thus the function

$$(4.28) \qquad E : x \mapsto -2x\big(g_0(x^2) + xg_1(x^2)\big),$$

considered up to composition by reversible odd functions (i.e. E is equivalent to $E \circ \psi$ for any ψ odd reversible) is a local invariant of X at p. We will call the equivalence $E \sim E \circ \psi$ the **odd left equivalence** class of E.

The local function

$$(4.29) \qquad E(x) = -2x\big(g_0(x^2) + xg_1(x^2)\big)$$

(considered up to odd left equivalence) will be called the **eigenvalue function** or also the **frequency function** of X at p. According to the above discussion and Proposition 4.2, this eigenvalue function is the full invariant of (X, F) at the generic nilpotent singular point p. In other words, we have proved the following theorem:

Theorem 4.3. *1) Let p be a generic nilpotent singularity of a smooth integrable system $(X, \mathcal{F})$. Then there is a local smooth coordinate system (x, y) in a neighborhood $\mathcal{U}$ of p in which X has the following normal form:*

$$(4.30) \qquad X = \big(g_0(y) + xg_1(y)\big)(y - x^2)\frac{\partial}{\partial x},$$

where g_0, g_1 are two smooth functions and $g(0) \neq 0$.

2) If $\widetilde{X} = \big(\widetilde{g}_0(y_1) + x_1\widetilde{g}_1(y_1)\big)(y_1 - x_1^2)\frac{\partial}{\partial x_1}$ is another generic nilpotent integrable smooth vector field in another smooth local normal form, then X and $\widetilde{X}$ are locally smoothly isomorphic if and only if their respective eigenvalue functions $E(x) = -2x\big(g_0(x^2) + xg_1(x^2)\big)$ and $\widetilde{E}(x_1) = -2x_1\big(\widetilde{g}_0(x_1^2) + x_1\widetilde{g}_1(x_1^2)\big)$ are locally odd left equivalent, i.e. there is a local smooth odd reversible function ψ such that $\widetilde{E}\big(\psi(x)\big) = E(x)$.

The eigenvalue function $E(x)$ in Formula (4.29) and Theorem 4.3 is a smooth function, but it is not very convenient for the semi-local and global study, because the variable x is not a first integral of the system. So instead of $E(x)$ we will consider the function

$$(4.31) \qquad F(y) = -2\sqrt{y}(g_0(y) + \sqrt{y}g_1(y)).$$

This function $F(y)$ is double-valued and not smooth in y (it is smooth only in $\sqrt{y}$), but since y is a first integral in the local normal form, we can project F to a double-valued function on the local base space. F will be called the

double-valued eigenvalue function. Another advantage of F is that, instead of odd left equivalence, we now have usual left equivalence, i.e. for any local smooth diffeomorphism $\phi : (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$, F is equivalent to $F \circ \phi$. The second part of Theorem 4.3 can be now restated as follows:

Theorem 4.4. *Two singularities of Type IV are smoothly locally isomorphic if and only if their corresponding double-valued eigenvalue functions are left equivalent.*

4.2. **Complexification and regularized monodromy.** Consider a generic nilpotent smooth integrable vector field X in normal form:

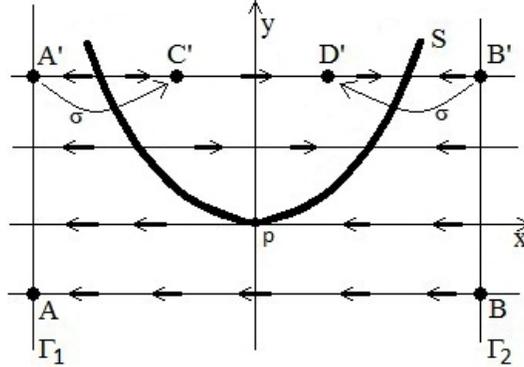$$(4.32) \qquad X = \big(g_0(y) + xg_1(y)\big)(y - x^2)\frac{\partial}{\partial x}.$$

FIGURE 7. Type IV singular point

Fix two local curves Γ_1 and Γ_2 which are transversal to the lines $\{y = constant\}$, and which lie on the two different sides of the nilpotent singular point p in the above coordinate system (x, y). For example, one can take $\Gamma_1 = \{x = -c\}$ and $\Gamma_2 = \{x = c\}$ for some small positive constant c.

For each point $A \in \Gamma_1$ there is a unique point $B \in \Gamma_2$ such that $y(A) = y(B)$, i.e. A and B lie on the same local smooth invariant curve of X.

If $y(A) < 0$ then locally the vector field X is non-singular on the invariant curve $\{y = y(A)\}$, and so there is a unique time T (if $g_0(0) > 0$ then $T < 0$, and if $g_0(0) < 0$ then $T > 0$) such that the time-T flow φ_X^T of X moves A to B

$$(4.33) \qquad \varphi_X^T(A) = B.$$

Of course, T depends on the choice of Γ_1, Γ_2 and the value of $y = y(A)$, so we can write it as a function of y:

$$(4.34) \qquad T = T_{\Gamma_1, \Gamma_2}(y).$$

It is clear that

$$\lim_{y \to 0^-} T_{\Gamma_1, \Gamma_2}(y) = \infty \tag{4.35}$$

because $X(p) = 0$, thus the function T_{Γ_1, Γ_2} is singular at $y = 0$. We want to regularize this function, i.e. write it as

$$T_{\Gamma_1, \Gamma_2}(y) = \mathcal{S}(y) + \widehat{T}_{\Gamma_1, \Gamma_2}(y) \tag{4.36}$$

where $\widehat{T}_{\Gamma_1, \Gamma_2}(y)$ is a smooth function, and $\mathcal{S}$ is a singular function which does not depend on the choice of Γ_1 and Γ_2.

If $y = y(A) > 0$ then the invariant curve $\{y = y(A)\}$ contains two singular points $(\pm\sqrt{y}, y)$, and we can't even go from B to A by the flow of X. But, as was shown in [25] and in Subsection 3.1, we can go from B to A using the flow φ_X by "jumping over the walls" as follow:
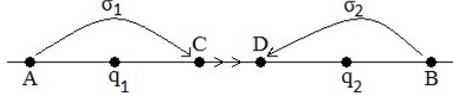

FIGURE 8. Jumping over the walls.

Denote by q_1 and q_2 the two nondegenerate hyperbolic singular points with eigenvalue 0 on the invariant curve $\{y = y(A) = y(B)\}$. Denote by σ_1 and σ_2 the two reflection maps associated to q_1 and q_2 respectively. Put $C = \sigma_1(A), D = \sigma_2(B)$, and define the regularized time function $\widehat{T}_{\Gamma_1, \Gamma_2}(y)$ when $h > 0$ by the formula

$$\varphi_X^{\widehat{T}_{\Gamma_1, \Gamma_2}(y)}(C) = D. \tag{4.37}$$

We will show that the singular function $\mathcal{S}(y)$ can be chosen in such a way that the regularized time function $\widehat{T}_{\Gamma_1, \Gamma_2}(y)$ for $y > 0$ agrees with $\widehat{T}_{\Gamma_1, \Gamma_2}(y)$ for $y < 0$ to become together a smooth function of y at $y = 0$.

For simplicity, let us first look at the analytic case, i.e. the case when the functions g_0 and g_1 in the normal form
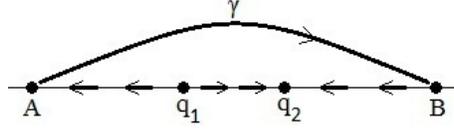
$$X = \big(g_0(y) + x g_1(y)\big)(y - x^2)\frac{\partial}{\partial x} \tag{4.38}$$

are real analytic functions. In this analytic case, we can use the complexification method to study the regularized time function.

By complexification, the analytic vector field in a neighborhood of 0 in $\mathbb{C}^2$. On each local invariant complex line $\mathcal{C}_y = \{y = \text{ constant }\}$, X vanishes at exactly two points $q_{1,2} = \pm\sqrt{y}$. In the complex line $\mathcal{C}_y$, we can go from A to B by a path γ which avoids q_1 and q_2 as shown in the Figure 9.

Then we can put $T^{\mathbb{C}}_{\Gamma_1, \Gamma_2}(y)$ equal to the complex time to go by φ_X along γ from A to B. The formula for $T^{\mathbb{C}}_{\Gamma_1, \Gamma_2}(y)$ is:

$$T^{\mathbb{C}}_{\Gamma_1, \Gamma_2}(y) = \int_\gamma \frac{dx}{\big(g_0(y) + x g_1(y)\big)(y - x^2)}. \tag{4.39}$$

FIGURE 9. Going from A to B using complex path γ.

Of course, the above formula depends only on the homotopy class of γ. When $y \in \mathbb{R}, y > 0$, we can also imagine γ as in Figure 10:
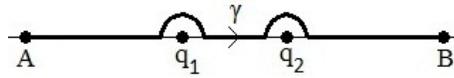

FIGURE 10. Going half-circle around q_1 and q_2.

The above path γ consist of 3 pieces of real time (on the real path from A to B) and two half-circles of pure imaginary time (going half-circle around q_1 and q_2). The total real time is actually equal to $\widehat{T}_{\Gamma_1,\Gamma_2}(y)$, while the imaginary time for going half-circle around each q_i is equal to $-i\pi$ divided by the eigenvalue of X at q_i. Thus we have

$$(4.40) \qquad \widehat{T}_{\Gamma_1,\Gamma_2}(y) = \Re\big(T^{\mathbb{C}}_{\Gamma_1,\Gamma_2}(y)\big)$$

where $\Re$ denotes the real part.

Similarly, when $y \in \mathbb{R}_-$, we can imagine the path γ as in Figure 11.
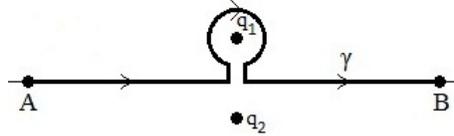
FIGURE 11. Going around q_1.

The time for going from A to B along γ is equal to the time going from A to B along the real path plus the time for going around the singular point $q_1 = (i\sqrt{-y}, y)$ in the negative direction of the complex plane. The time for going around q_1 in the positive direction is equal to $2\pi\sqrt{-1}$ divided by the eigenvalue of X at q_1, so it is a complex number whose real part is (for $y < 0$):

$$(4.41) \qquad S(y) = \frac{\pi\sqrt{-y}g_0(y)}{-y(g_0^2(y) - yg_1^2(y))}.$$

Thus we can put, for $y \leq 0$:

$$(4.42) \qquad \widehat{T}_{\Gamma_1,\Gamma_2}(y) = T_{\Gamma_1,\Gamma_2}(y) - \frac{\pi\sqrt{-y}g_0(y)}{-y(g_0^2(y) - yg_1^2(y))}.$$

Formula (4.42) makes sense also in the smooth non-analytic case, and together with Formula (4.37) gives us a local smooth function in y. This local smooth function given by Formula (4.37) for $y > 0$ and by Formula (4.42) for $y \leq 0$ is called the **regularized time function** for going from Γ_1 to Γ_2 by the flow of X.

The time function $\widehat{T}_{\Gamma_1,\Gamma_2}$ depends on Γ_1 and Γ_2. In order to make it into something independent of Γ_1 and Γ_2, we go back by the flow of X from Γ_2 to Γ_1, not in the same way, but in the other way "around the globe" to make a loop (a closed level set).

Assume, for simplicity, that the level set $\mathcal{N}$ of the Type IV singular point p does not contain any singular point of Type III, and it does not contain any other Type IV singular point either, and moreover the neighborhood N in the surface Σ is orientable. Then, similarly to the case of Type II level sets, it is easy to see that $\mathcal{N}$ is a regular level set of the associated fibration. Denote by

$$(4.43) \qquad\qquad R_{\Gamma_2,\Gamma_1}(y)$$

the time function for returning from Γ_2 to Γ_1 "by going around the globe", i.e. not by the previous path γ, but by the complementary path in each level set. Of course, if that complementary path crosses some Type II singular points, then we will jump over them as we did before, and the time function is still well defined. The function

$$(4.44) \qquad\qquad \widehat{M}(y) = \widehat{T}_{\Gamma_1,\Gamma_2}(y) + R_{\Gamma_2,\Gamma_1}(y)$$

is called the **regularized monodromy function** near the level set $\mathcal{N}$.

Observe that, when $y > 0$ then $\widehat{M}(y) = M(y)$ is the usual monodromy of a regular level set with Type II singular points, so the function $\widehat{M}(y)$ for $y \geq 0$ is, up to left equivalence, a semi-local invariant of the system. On the other hand, similarly to the case of Type III level sets, $\widehat{M}(y)$ for $y < 0$ considered up to left equivalence is NOT an invariant, only its Taylor series at $y = 0$ is. But this Taylor series is also determined by $\widehat{M}(y)$ for $y > 0$, so we can actually forget about $\widehat{M}(y)$ for $y < 0$ in the semi-local classification of Type IV level sets. The function $\widehat{M}(y)$ for $y \geq 0$ will be called the **truncated monodromy function** (we truncated the part with $y < 0$). Summarizing, we have:

Theorem 4.5. *Let p be a Type 4 singular point of a smooth integrable system $(X, \mathcal{F})$ on a compact surface Σ, such that the level set $\mathcal{N}$ of p does ot contain any other singular point of Type III or Type IV, and the surface Σ is orientable near $\mathcal{N}$. Then $\mathcal{N}$ is a regular level set of the associated fibration. Moreover, the local eigenvalue functions in a neighborhood $\mathcal{U}(N)$ of N and the truncated monodromy function in $\mathcal{U}(N)$, considered as functions on the local base space andup to simultaneous local left equivalence, classify $(X, \mathcal{F})$ up to semi-local smooth isomorphisms.*

The case when $\mathcal{U}(N)$ is no-orientable can be classified similarly: in this case it is the Taylor series of the regularized monodromy function which is the continuous semi-local invariant (in additional to the local invariants). The case when $\mathcal{N}$ contains other singular points of Type IV and Type III is more complicated: in this case, instead of the regularized monodromy, we have to talk about regularized period cocycles, similarly to the case of level sets of Type III. We don't want to go into the details here.

5. Global classification

In order to obtain a global classification of integrable systems $(X, \mathcal{F})$, we just need to collect all the local and semi-local invariants. (There is no specifically global invariant, due to the fact that the base space of the associated singular fibration has only 1 dimension and does not carry any additional structure on it besides the smooth structure). We can formulate the following classification theorems, whose proof is simply a combination of the results of the previous sections:

Theorem 5.1. *Two smooth integrable systems $(X_1, \mathcal{F}_2)$ and $(X_2, \mathcal{F}_2)$ on closed surfaces Σ_1 and Σ_2 respectively are smoothly orbitally equivalent if any only if there is a homeomorphism $\Phi : \Sigma_1 \to \Sigma_2$ which is a bijection when restricted to the sets of singular points of Type k of X_1 and X_2 for each $k = I, II, III, IV$, and such that for each singular point $p \in \Sigma_1$ of Type III the ratio of the two eigenvalues of X_1 at p_1 is equal to the ratio of the two eiganvalues of X_2 at $\Phi(p_1)$.*

Theorem 5.2. *Two smooth integrable systems $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ on closed surfaces Σ_1 and Σ_2 respectively are smoothly isomorphic if any only if there is a smooth diffeomorphism $\Phi : \Sigma_1 \to \Sigma_2$ which is a smooth orbital equivalence of the two systems, such that the quotient map $\widehat{\Phi} : \mathcal{B}_1 \to \mathcal{B}_2$ from the quotient space $\mathcal{B}_1$ to the quotient space $\mathcal{B}_2$ is a simultaneous left equivalence between the following objects of $(X_1, \mathcal{F}_1)$ (considered as functions on $\mathcal{B}_1$) and the corresponding objects of $(X_2, \mathcal{F}_2)$: the period functions and the monodromy functions (for regular level sets which may contain Type II singular points, and also near Type I singular points), the eigenvalue functions (for Type II singularities), the truncated monodromy functions (for simple Type IV level sets), the Taylor series of the regularized monodromy function (for twisted Type IV level sets), the Puiseux series of the frequency functions and the cohomology classes of the period cocycles (for Type III level sets and also for mixed Type III - Type IV level sets).*

6. Hamiltonianization

We say that a vector field X on a manifold M is **Hamiltonianizable**, or that it admits a **hamiltonianization**, if there exists a function H and a Poisson structure Π on M such that $X = X_H := dH \lrcorner \Pi$. We will distinguish the symplectic case (when Π is nondegenerate) from the degenerate case (when Π vanishes at some points).

Theorem 6.1. *Let $(X, \mathcal{F})$ be a weakly nondegenerate smooth integrable system on a compact surface Σ. Then X is Hamiltonianizable by a symplectic structure if and only if the following conditions are satisfied:*
i) Every singular point of Type III is traceless, i.e. the sum of the two eigenvalues of X at that point is 0,
ii) There is a global smooth coordinate function on the base space $\mathcal{B}$,
iii) The surface Σ is orientable,
iv) $(X, \mathcal{F})$ does not contain singular points of Type II and Type IV.

Proof. It is clear that the above conditions are necessary for X to be Hamiltonianized, because symplectic manifolds are orientable, symplectic vector fields have zero trace at singular points, and the Hamiltonian function of the system will project to a global coordinate function on the base space. Let us show that these conditions are also sufficent.

Condition i) implies that X can be written as $X = g(xy)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ near each singular point of Type III. According to Proposition 3.5, near every hyperbolic level set $\mathcal{N}$ (i.e. a level set which contains a singular point of Type III) there is a first integral $F_{\mathcal{N}}$ without multiplicity at $\mathcal{N}$, i.e. the order of vanishing of $F_{\mathcal{N}}$ at every component of $\mathcal{N}$ is 1, or in the other words, all singular points of F in $\mathcal{U}(\mathcal{N})$ are nondegenerate. According to local geometric linearization theorem , near an elliptic singular point there is also a nondegenerate first integral of type $F = x^2 + y^2$ (see Section 2). Conditions ii) and iii) then implies that there is a global first integral H which is a Morse function on Σ.

The symplectic form can be chosen locally in such a way that $X = X_H$, i.e. X is the Hamiltonian vector field of the above Morse first integral H:

a) In a neighborhood $\mathcal{U}(p_i)$ of an elliptic singular point p_i where $X = g(x^2 + y^2)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ and $H = h(x^2 + y^2)$, put

$$(6.1) \qquad w_i = -\frac{g(x^2 + y^2)}{2h'(x^2 + y^2)} dx \wedge dy.$$

where h' is the derived function of h.

b) In a neighborhood $\mathcal{U}(q_j)$ of a hyperbolic singular point q_j where $X = g(xy)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ and $H = h(xy)$, put

$$(6.2) \qquad w_j = -\frac{g(xy)}{h'(xy)} dx \wedge dy.$$

c) In a neighborhood $\mathcal{U}(r_k)$ of a regular point r_k where $X = f(y)\frac{\partial}{\partial x}$ and $H = h(y)$ put

$$(6.3) \qquad w_k = \frac{f(y)}{h'(y)} dx \wedge dy.$$

We can choose the neighborhoods $\mathcal{U}(p_i), \mathcal{U}(q_j), \mathcal{U}(r_k)$ so that they form a finite open covering of Σ, and the local canonical coordinate systems in

them so that the above symplectic form w_i, w_j, w_k give the same orientation of Σ. Let

$$(6.4) \qquad 1 = \sum_i \psi_i + \sum_j \psi_j + \sum_k \psi_k$$

be a partition of unity on Σ such that $\psi_i \geq 0$ everywhere and $\psi_i = 0$ outside of $\mathcal{U}(p_i)$, and similarly for ψ_j and ψ_k. Put

$$(6.5) \qquad w = \sum_i \psi_i w_i + \sum_j \psi_j w_j + \sum_k \psi_k w_k.$$

Then w is a global symplectic form on Σ, and we have

$$(6.6) \qquad X = X_H$$

with respect to w on Σ. $\hfill\square$

Theorem 6.2. *Let $(x, \mathcal{F})$ be a weakly nondegenerate smooth integrable system on a compact surface Σ. Then X is Hamiltonianizable by a Poisson structure if and only if the following conditions are satisfied:*
i) Every singular point of Type III is traceless, i.e. the sum of the two eigenvalues of X at that point is 0,
ii) There is a global smooth coordinate function on the base space $\mathcal{B}$,
iii) Σ is orientable in the neighborhood of every level set.

Remark 6.3. The first two conditions in Theorem 6.2 are the same as in Theorem 6.1, but the last condition in Theorem 6.2 is much weaker than the last two conditions in Theorem 6.1: in the Poisson case we allow Type II and Type IV singular points (the set of such points is a disjoint union of closed simple curves on Σ), and Σ is not required to be orientable globally: it can be non-orientable in the neighborhood of a closed curve of singular points of Type II and Type IV. See Figure 12 for an example.
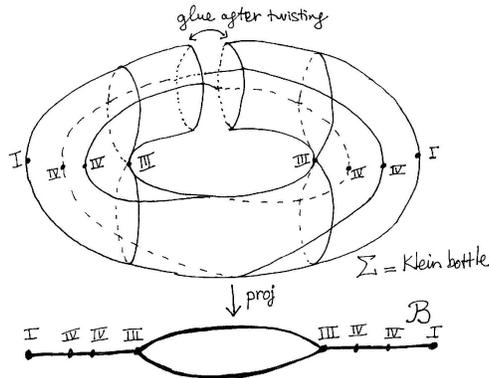

FIGURE 12. A Hamiltonian system on an non-orientable Poisson surface

Proof. The proof is similar to the proof of Theorem 6.1. We still have a global Morse first integral. We can still construct Poisson structures locally, and then sum them up together by a partition of unity on Σ. Near a Type II singular point where

$$(6.7) \qquad X = \gamma(y)x\frac{\partial}{\partial x} \text{ and } H = h(y)$$

we can choose the Poisson structure

$$(6.8) \qquad \Pi = \frac{\gamma(y)}{h'(y)}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Near a Type IV singular point where

$$(6.9) \qquad X = (y + G(x,y))\frac{\partial}{\partial x} \text{ and } H = h(y)$$

we can choose the Poisson structure

$$(6.10) \qquad \Pi = \frac{y + G(x,y)}{h'(y)}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The rest of the proof is straightforward. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.4. The Poisson structure in Theorem 6.2 vanishes on the set of singular points of Type II and Type IV. This set is a disjoint union of regular simple closed curves, and the Poisson structure is locally isomorphic to $x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ near every of these singular points. Such Poisson structures are stable under perturbations (see, e.g., [8, 19]), and generic Hamiltonian systems on them will admit singularities of Type II and Type IV. This is one more good reason to include nilpotent Type IV singularities in our definition of weaky-nondegenerate integrable systems on surfaces.

References

[1] M. Ayoul, N.T. Zung, *Galoisian obstructions to non-Hamiltonian integrability*, Comptes Rendus Mathématiques, 348 (2010), Issue 23, 1323-1326.

[2] A.V. Bolsinov, *A smooth trajectory classification of integrable Hamiltonian systems with two degrees of freedom*, Sbornik Math. 186 (1995), No. 1, pp. 1-27.

[3] A.V. Bolsinov, A.T. Fomenko, Integrable Hamiltonian Systems: Geometry, Topology, Classification, Chapman & Hall/CRC, 2004, xvi+730 pp.

[4] A.V. Bolsinov , Vu Ngoc San, *Symplectic equivalence for integrable systems with common action integrals*, in preparation.

[5] K.T. Chen, *Equivalence and decomposition of vector fields about an elementary critical point*, Amer. J. Math., 85 (1963), 693–722.

[6] Y. Colin de Verdière, J. Vey, *Le lemme de Morse isochore*, Topology 18 (1979), no. 4, 283–293.

[7] Jean-Paul Dufour, Pierre Molino, Anne Toulet, *Classification des systèmes intégrables en dimension 2 et invariants des modèles de Fomenko*, C. R. Acad. Sci. Paris Sér. I Math. 318 (1994), no. 10, 949–952.

[8] Jean-Paul Dufour, Nguyen Tien Zung, Poisson structures and their normal forms, Progress in Mathematics, 242. Birkhäuser Verlag, Basel, 2005.

[9] H. R. Dullin and S. Vũ Ngọc, *Symplectic invariants near hyperbolic-hyperbolic points*, Regul. Chaotic Dyn. 12 (2007), no. 6, 689–716.

[10] A.T. Fomenko, *The topology of surfaces of constant energy of integrable Hamiltonian systems and obstructions to integrability*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 6, 1276–1307, 1344.

[11] Jaume Giné, Jaume Llibre, *On the planar integrable differential systems*, Z. Angew. Math. Phys. 62 (2011), no. 4, 567–574.

[12] Xianghong Gong, *Integrable analytic vector fields with a nilpotent linear part*, Ann. Inst. Fourier (Grenoble) 45 (1995), no. 5, 1449–1470.

[13] André Haefliger, Georges Reeb, *Variétés (non séparées) à une dimension et structures feuilletées du plan*, Enseignement Math. (2) 3 (1957), 107–125.

[14] B.S. Kruglikov, *Exact smooth classification of Hamiltonian vector fields on two-dimensional manifolds*, Math. Notes, 61 (1997), no.2, 146-163.

[15] B.S. Kruglikov, *Exact classification of nondegenerate devergence-free vector fields on surfaces of small genus*, Mathematical Notes, Volume 65 (1999), Number 3, 280-294.

[16] S. I. Maksymenko, *Symmetries of center singularities of plane vector fields*, Nonlinear Oscil. (N. Y.) 13 (2010), no. 2, 196–227.

[17] J. Moser, *On the volume elements on a manifold*, Trans. Am. Math. Soc. 120 (1965), 286–294.

[18] A. A. Oshemkov, *Morse functions on two-dimensional surfaces. Coding of singularities*, Proc. Steklov Inst. Math. 1995, no. 4 (205), 119-127.

[19] O. Radko, *A classification of topologically stable Poisson structures on a compact oriented surface* J. Symplectic Geometry, 1 (2002), no. 3, 523-542

[20] S. Sternberg, *On the structure of local homeomorphisms of Euclidean n-space*, II, Amer. J. of Math., 80 (1958), 623-631.

[21] Floris Takens, *Singularities of vector fields*, Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 47–100.

[22] S. Vũ Ngọc, *On semi-global invariants for focus-focus singularities*, Topology, 42(2003), No. 2, 365–380.

[23] N.T. Zung, *Convergence versus integrability in Poincaré-Dulac normal form*, Math. Res. Lett. 9 (2002), no. 2-3, 217-228.

[24] N.T. Zung, *Actions toriques et groupes d'automorphismes de singularités des systèmes dynamiques intégrables*, Comptes Rendus Mathématiques 336 (2003), Issue 12, pages 1015-1020.

[25] N.T. Zung, *Nondegenerate singularities of integrable dynamical systems*, preprint arXiv:1108.3551v2 (2012).

[26] N.T. Zung, *Linearzation of nondegenerate singularities of smooth integrable dynamical systems*, in preparation (2012).

[27] N.T. Zung, N. V. Minh, *Geometry of nondegenerate $\mathbb{R}^n$-actions on n-manifolds*, preprint arXiv:1203.2765 (2012).

Institut de Mathématiques de Toulouse, UMR5219, Université Toulouse 3

E-mail address: `tienzung.nguyen@math.univ-toulouse.fr, minh@math.univ-toulouse.fr`