Convexity of integrable Hamiltonian systems with focus-focus singularities

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Outline of the talk

1. Symplectic toric convexity
2. Symplectic convexity: What, why, and what for?
3. Integrable Hamiltonian systems and their singularities
4. Toric-focus systems
5. Affine structure on the base space
6. Local and global convexity
7. Integral affine black holes
8. Positive results on convexity with monodromy

Convexity of Hamiltonian torus actions

• **Symplectic manifold** (M, ω) means that ω is a nondegenerate closed 2-form on M.

• **Hamiltonian vector field** X_f of a function f on (M, ω) defined by:

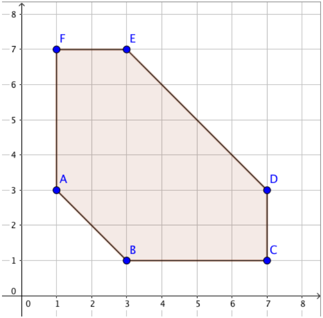
$$X_f \lrcorner \omega = -df$$

• **Momentum map** $J = (J_1, \ldots, J_k) : (M, \omega) \to \mathbb{R}^k$ of a **Hamiltonian torus** $\mathbb{T}^k$**- action** means that $(X_{J_1}, \ldots, X_{J_k})$ are generators of a $\mathbb{T}^k$-action on M.

Theorem (Atiyah 1982, Guillemin–Sternberg 1982)

Let (M^{2n}, ω) be a 2n-dimensional symplectic manifold endowed with a Hamiltonian $\mathbb{T}^k$-action with momentum map $J : M \to \mathbb{R}^k$. Then:
i) The fibers of J are connected;
ii) The momentum map J is open onto its image;
iii) $J(M)$ is a compact convex polytope, namely the convex hull of the image of the fixed point set of the $\mathbb{T}^k$-action.

Some particular cases



• Schur (1923) - Horn (1954):
The set of diagonals of an isospectral set of Hermitian $n \times n$ matrices, viewed as a subset of $\mathbb{R}^n$, is equal to the convex polytope whose vertices vertices are the vectors formed by the $n!$ permutations of its eigenvalues.

• Kostant (1973): The projection of a coadjoint orbit of a connected compact Lie group relative to a bi-invariant inner product onto the dual of a Cartan subalgebra is the convex hull of an orbit of the Weyl group.

• Delzant (1988): 1-1 correspondence between compact **symplectic toric manifolds** (the case of $\mathbb{T}^n$- actions on $2n$-dimensional manifolds) and **Delzant polytopes** (convex n-dimensional polytopes which are simple, rational and regular)

Generalizations

See Overview in our preprint (Ratiu-Wacheux-Zung 2017)

• **non-Abelian** group actions (Kirwan 1984); **symplectic orbifolds** (Lerman–Meinrenken–Tolman–Woodward, 1997-98); **toric degenerations**, e.g. Gelfand-Cetlin system on $su(n)^*$.

• **non-compact proper actions**, **nonlinear convexity**, **proper groupoids**, etc.: Kostant (1973), Hilgert, Neeb, and Plank (1994), Lerman (1995), Heinzner–Huckleberry (1996), Weinstein (2001), Zung (2006), etc.

•**infinite dimensions** : Loop groups (Atiyah–Pressley 1983), Kac–Moody groups , Kac–Peterson 1984), Banach–Lie groups of operators on a separable Hilbert space (Neumann 1999, 2002), area-preserving diffeomorphisms on an annulus (Bloch–Flashka–Ratiu 1993), etc.

• **Presymplectic** instead of symplectic: Lin-Sjamaar (2017), Ratiu–Zung (2017), see the talk by Tudor Ratiu.

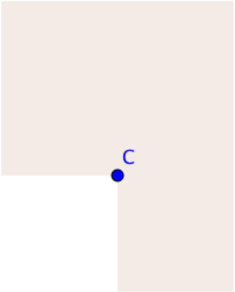
This talk: Generalization to the case of **toric-focus** integrable Hamiltonian systems.

Symplectic convexity: What, why, and what for?

• *What*: Convexity of the (intrinsic, transverse) **affine structure** on the **base space** of a fibration given by a momentum map.

• *Why*: **Local convexity** via **normal forms**, plus **local-to-global convexity principle**, which goes back to the following simple theorem of Tietze (1928) and Nakajima (1928): *A* **closed** *set* C *in* $\mathbb{R}^n$ *is* **convex** *if and only if it is* **connected** *and* **locally convex**.

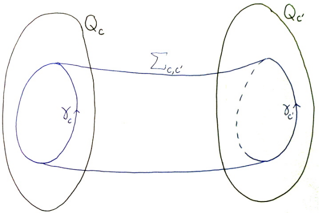
Local convexity = every point admits a convex neighborhood. Without closedness, the theorem is false. E.g., the figure without point C is non-convex but locally convex.



• *What for*: Many possible implications of convexity in combinatorics (counting points, volume etc.), topology (Morse theory, computation of cohomology, topological onstructions), optimization, etc. Even if we donn't know any application, it's still an interesting curioisty.

Where does the affine structure come from?

Affine functions = action functions = period integrals over generating 1-form α or (twisted/pre) symplectic 2-form ω



In the picture: $Q_c, Q_{c'}$ are level sets (of momentum map or singular fibration), $\gamma_c, \gamma_{c'}$ are 1-cycles with homotopy cylinder $\Sigma_{c,c'}$. Affine function:

$$F(c) = \int_{\Sigma_{c,c'}} \omega = \int_{\gamma_c} \alpha - \int_{\gamma_{c'}} \alpha$$

This is called **Action function formula**: Einstein (1917, Bohr-Sommerfeld quantization), Mineur (1935, proof of action-angle variables), Arnold, ...

Local-global convexity principle

- Tietze–Nakajima local-global convexity principle admits many versions and generalizations over the last century, also in infinite dimensions.
- Condeveaux–Dazord–Molino (1988) were the first to use it (instead of Morse theory) to give an elegent simple proof of symplectic convexity theorems.
- Hilgert–Neeb–Planck (1994) gave a version of it well adapted for symplectic convexity. Since then, it became a very important tool in convexity. In particular, Flashka–Ratiu (1996) needed it to prove convexity for compact Poisson Lie groups (Morse theory didn't work there).
- The following simple version also works very well for symplectic convexity:

Lemma (Local-global convexity lemma, Z 2006)

Let X be a connected locally convex regular affine manifold with boundary, and $\phi : X \to \mathbb{R}^m$ a proper locally injective affine map. Then ϕ is injective and its image $\phi(X)$ is convex in $\mathbb{R}^m$.

Integrable Hamiltonian systems in the sense of Liouville

- $H : (M, \omega) \to \mathbb{R}$: a Hamiltonian function on a symplectic manifold of dimension $2n$ ($n \geq 1$ is called the **degree of freedom**).
- $H_1 = H$ is automatically a first integral of the system. Integrability a la Liouville means that there exist $n-1$ additional commuting first integrals $H_2, \ldots, H_n$ such that the **momentum map**

$$\mathbf{H} = (H_1, \ldots, H_n) : M^{2n} \to \mathbb{R}^n$$

is of rank n (i.e. the functions $H_1, \ldots, H_n$ are independent) a.e. We will assume that the map $\mathbf{H}$ is **proper**.
- **Base space** $\mathcal{B} = \{$ connected level sets of the momentum map $\mathbf{H} = (H_1, \ldots, H_n) : M \to \mathbb{R}^n\}$. $\mathcal{B}$ admits a natural singular **integral affine structure**, due to the existence of *action-angle variables*

$$(D^n \times \mathbb{T}^n, \omega = \sum dp_i \wedge dq_i)$$

(Arnold-Liouville-Mineur theorem, proved by Mineur in 1935).
- Problem: what about the **convexity of** $\mathcal{B}$?

Singularities of integrable Hamiltonian systems

Most singular points of the momentum map $\mathbf{H} : M^{2n} \to \mathbb{R}^n$ are **nondegenerate**; they can be linearized locally (Williamson, Rüssmann, Vey, Eliasson) or near a compact orbit (Miranda–Zung).

Theorem (Local linearization, Vey–Eliasson)

If $p \in M^{2n}$ is a non-degenerate singular point of an integrable Hamiltonian system $F = (F_1, \ldots, F_n) : M \to \mathbb{R}^n$, then $\exists$ local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about p, such that $\{F_i, q_j\} = 0$, for all i, j, where

- $q_i = \mathrm{e}_i = x_i^2 + \xi_i^2$ $(1 \le i \le k_e)$ *are elliptic components,*
- $q_{k_e+i} = \mathrm{h}_i = x_{i+k_e}\xi_{i+k_i}$ $(1 \le i \le k_h)$ *are hyperbolic components,*
- $\begin{cases} q_{2i-1+k_e+k_h} = \mathrm{f}_i^1 = x_{2i-1+k_e+k_h}\xi_{2i+k_e+k_h} - x_{2i+k_e+k_h}\xi_{2i-1+k_e+k_h} \\ q_{2i+k_e+k_h} = \mathrm{f}_i^2 = x_{2i-1+k_e+k_h}\xi_{2i-1+k_e+k_h} + x_{2i+k_e+k_h}\xi_{2i+k_e+k_h} \end{cases}$
  $(1 \le i \le k_f)$ *are focus-focus components,*
- $q_{k+i} = x_i$ $(1 \le i \le n - \kappa)$ *are regular components.*

Semi-local structure of nondegenerate singularities

Theorem (Zung 1996)

N = non-degenerate singular fiber of corank κ and Williamson type $\Bbbk = (k_e, k_h, k_f)$ in an integrable Hamiltonian system given by a proper momentum map $\mathbf{H} : M^{2n} \to \mathbb{R}^n$. Then $\exists$ neighborhood $\mathcal{U}(N)$ of N in M^{2n}, saturated by the fibers of the system, such that:
(i) *$\exists$ an effective Hamiltonian action of $\mathbb{T}^{k_e+k_f+(n-\kappa)}$ on $\mathcal{U}(N)$ which preserves the system. This number $k_e + k_f + (n-\kappa)$ is maximal possible.*
(ii) *$(\mathcal{U}(N)$, associated Lagrangian torus fibration) is homeomorphic to the quotient of a direct product of elementary non-degenerate singularities and a regular Lagrangian torus foliation of the type*

$$(\mathcal{U}(\mathbb{T}^{n-\kappa}), \mathcal{L}^r) \times (P^2(N_1^e), \mathcal{L}_1^e) \times \cdots \times (P^2(N_{k_e}^e, \mathcal{L}_{k_e}^e) \times$$
$$\times (P_h^2(N_1^h, \mathcal{L}_1^h) \times \cdots \times (P^2(N_{k_h}^h), \mathcal{L}_{k_h}^h) \times (P^4(N_1^f), \mathcal{L}_1^f) \times \cdots \times (P^4(N_{k_f}^f), \mathcal{L}_{k_f}^f)$$

by a free action of a finite group Γ.

Toric-focus systems

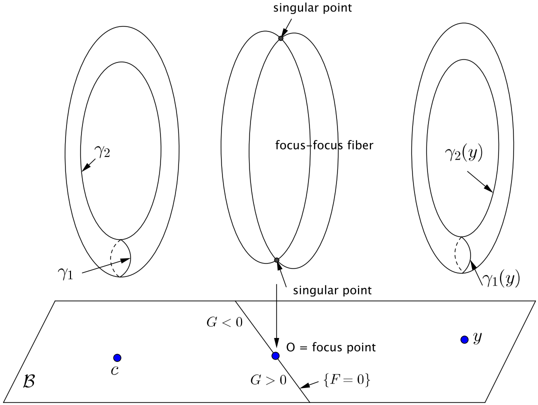
A **toric-focus system** is an integrable Hamiltonian system whose singularities are nondegenerate and have no hyperbolic component, only elliptic and/or focus-focus components. *Why toric-focus?*

- Can be found everywhere in physical systems, e.g.: spherical pendulum, Lagrange top, spin systems, focusing NLS equation, Jaynes–Cummings–Gaudin, etc. Related also to Lagrangian fibrations on Calabi–Yau (mirror symmetry), tropical affine structures.
- Base spaces are still manifolds (with hyperbolic singularities the base spaces are not manifolds). The integral affine structure has **focus singularities**, but one can still talk about convexity.
- Studied by many people. Duistermaat, Bates–Cushman, Knörrer, Horozov, Matveev, Audin, Babelon, Sadovskii-Zhilinskii, Hanssmann–Broer, Leung–Symington, Sepe–Hohloch–Sabatini, Vu Ngoc–Pelayo–Ratiu, Wacheux, Bolsinov, Isozimov, etc.
- Special cases: semi-toric (additional condition on torus actions).

Monodromy formula around focus singularities

$$\begin{pmatrix}[\gamma_1^{new}]\\ [\gamma_2^{new}]\end{pmatrix} = \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\begin{pmatrix}[\gamma_1]\\ [\gamma_2]\end{pmatrix}.$$

k = **index** of the focus singularity (the case with only 1 focus component)



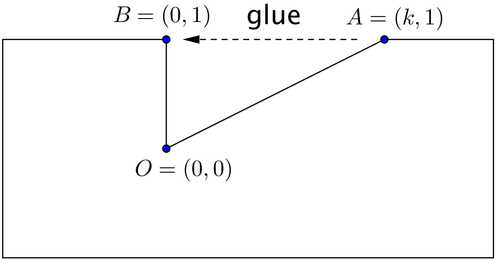
The base space of a torus-focus system

• *proj* : $M^{2n} \to \mathcal{B}$ - projection to the base space
• **Topology** on $\mathcal{B}$: Hausdorff, induced from M^{2n}
• **Differential structure** on $\mathcal{B}$ (sheaf of smooth functions): also induced from M^{2n}. With this differential structure, $\mathcal{B}$ is a smooth manifold with boundary and corners. (Focus points are regular w.r.t. the smooth structure)
• **Integral affine structure** on $\mathcal{B}$ is **singular**. (Local integral affine functions are those whose pull-back to M^{2n} give rise to Hamiltonian $\mathbb{T}^1$-actions). Focus points are singular points for the singular affine structure on $\mathcal{B}$.

Focus singularity on the base space

The 2D case: Multi-valued affine coordinate system (F, G): F is single-valued, G has 2 branches G_l and G_r:

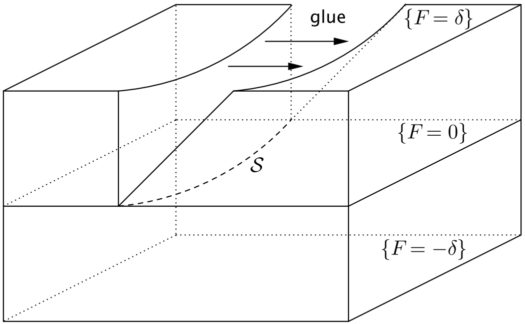
$$G_r = G_l + kF \text{ when } F > 0; \quad G_r = G_l \text{ when } F \leq 0.$$



Related to Duistermaat–Heckman formula w.r.t. Hamiltonian $\mathbb{T}^1$ near a focus-focus singular fiber. Convex because $k > 0$ (would be non-convex if $k < 0$)

Focus singularity on the base space

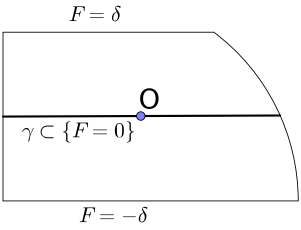
The higher-dimensional case (with 1 focus-focus component):

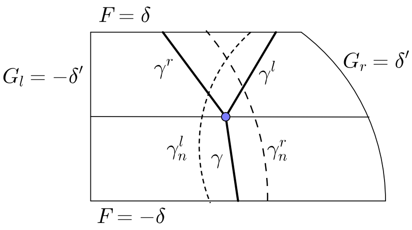


Focus submanifold S of codimension 2 is curved in general but lies on a flat $(n-1)$-dimensional subspace.

Straight lines and convexity

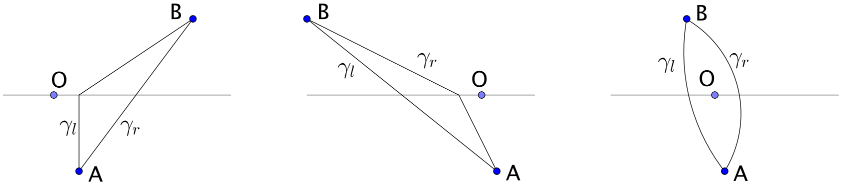
- We will say that our integral affine structure is **convex** if between any two points there is a **straight line** joining them.
- Problems due to focus singularities:
  - Straight lines may be non-unique, or may be non-existent (even when the base space is homeomorphic to a ball)
  - A straight line may be singular (it goes through focus singularities): extension problem when hitting a focus point
- Singular straight line = limit of a family of regular straight lines.
- **Branching** at focus points. Up to 2^k branches if k focus components.





Local convexity near a focus singularity

Two potential straight lines γ_l and γ_r from A to B (which might be broken).



The equations for the points of $\gamma_l : [0,1] \to Box$ are

$$F(\gamma_l(t)) = tF(B) + (1-t)F(A)\ ;\ G_l(\gamma_l(t)) = tG_l(B) + (1-t)G_l(A)$$

and the equations for the points of $\gamma_r : [0,1] \to Box$ are

$$F(\gamma_r(t)) = tF(B) + (1-t)F(A)\ ;\ G_r(\gamma_r(t)) = tG_r(B) + (1-t)G_r(A)$$

At least one of the two γ_l and γ_r is not broken

- The same situation in higher dimensions with only 1 focus component.

Local-global convexity principle

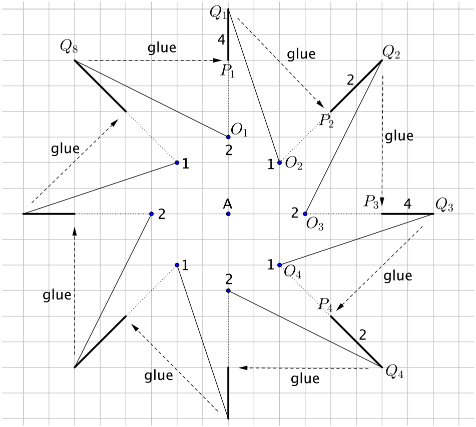
• Local-global convexity principle in the presense of focus singuarities is still OK if there is a global $\mathbb{T}^{n-1}$-action (the case when the monodromy is "additive", Abelian).
• Things may go wrong when the monodromy is complicated: The local-global convexity principle does not hold in general.

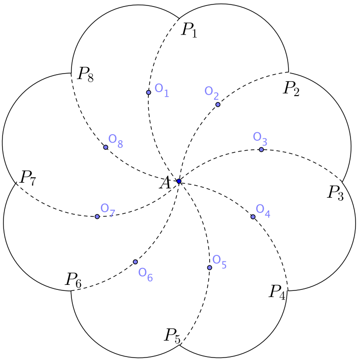
- Existence of non-convex integral affine S^2 (which is ocally convex).
- Non-convexity examples near focus^2 points.
- Non-convexity examples in 3D with two focus curves.

• Under some natural additional conditions, there are still positive global convexity results

Integral affine black-hole and non-convex S^2

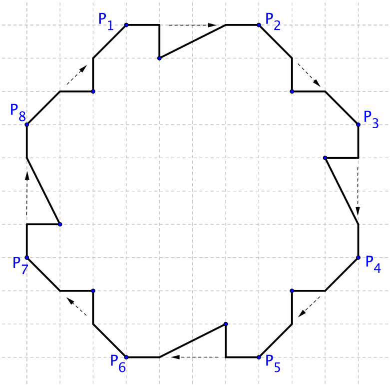
We have metric black holes (well known), and also affine black holes (shown below). Duality / mirror symmetry ?!
Glueing a shuriken into a flower. This flower is an "affine blackhole": the rays from the center A cannot get out of the folower.





Integral affine black-hole and non-convex S^2

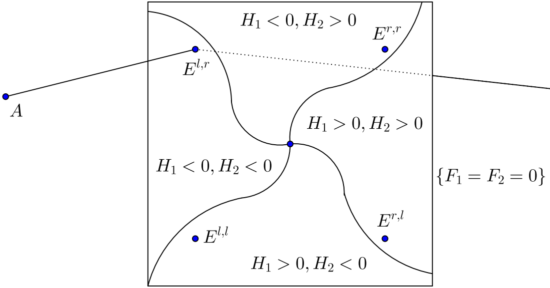
Glue the blackhole flower with an appropriate convex octagon to get a non-convex S^2;



Remark: There are also examples of *convex* integral affine S^2. So monodromy may lead to non-convexity but is not a total obstruction.

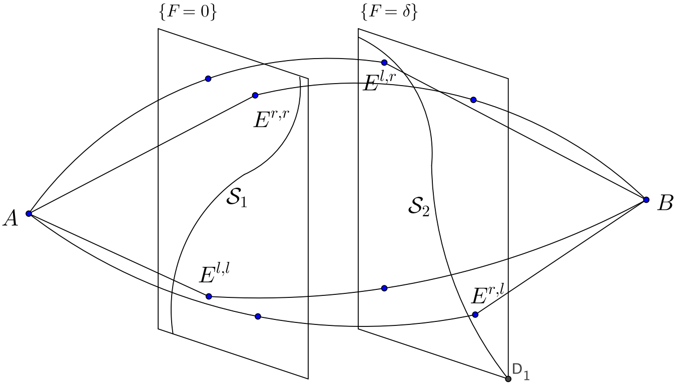
A non-convex 4D situation with a focus^2 point

In the following picture, all the 4 potential straight lines from A to B turn out to be broken lines, so there is no straight line from A to B.
Local coordinate system (F_1, H_1, F_2, H_2), where F_1, F_2 are integral affine.



A non-convex 3D situation with 2 focus curves

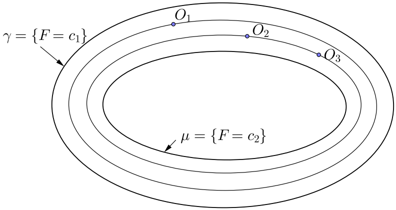
In the following picture, all the 4 potential straight lines from A to B also turn out to be broken lines, so there is no straight line from A to B.



Global convexity of 2D focus-toric systems: compact case

Theorem (Ratiu–Wacheux–Z 2017)

$\mathcal{B}$ = 2D locally-convex base space with non-empty boundary of a toric-focus system on a connected compact symplectic M^4 (with or without boundary). Then $\mathcal{B}$ is convex. Moreover, if $\mathcal{B}$ is orientable, then it is a disk or an annulus. If $\mathcal{B}$ is an annulus, there is a global single-valued non-constant affine function F on $\mathcal{B}$ such that of $\mathcal{B}$ and the boundary components of $\mathcal{B}$ are straight curves on which F is constant.



(Compare with Zung's thesis 1994, Leung–Symington 2010)

Global convexity of 2D focus-toric systems: proper case

Theorem

Let $\mathcal{B}$ be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, non-compact, symplectic, 4-manifold without boundary. Assume:

(i) *the system has elliptic singularities (i.e., the boundary of $\mathcal{B}$ is not empty);*

(ii) *the number of focus points in $\mathcal{B}$ is finite and the interior of $\mathcal{B}$ is homeomorphic to an open disk;*

(iii) *$\mathcal{B}$ is proper.*

Then $\mathcal{B}$ is convex (in its own underlying affine structure).

Remark: Without the properness condition the theorem would fail (Pelayo–Ratiu–Vu Ngoc: cartography of different proper and non-proper semi-toric systems).

Global convexity of semi-toric systems in higher dimensions

Theorem

Let $\mathcal{B}$ be the base space of a toric-focus integrable Hamiltonian system with n degrees of freedom on a connected compact symplectic manifold M. Assume that the system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action. Then $\mathcal{B}$ is convex.

Theorem

Let $\mathcal{B}$ be the n-dimensional base space of a toric-focus system on a connected, non-compact, symplectic, $2n$-manifold without boundary s.t.:

(i) *The system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action;*

(ii) *the set of focus points in $\mathcal{B}$ is compact;*

(iii) *the interior of $\mathcal{B}$ is homeomorphic to an open ball in $\mathbb{R}^n$;*

(iv) *$\mathcal{B}$ is proper.*

Then $\mathcal{B}$ is convex (in its own underlying affine structure).

THANK YOU!